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Kostka matrices at the level of bases and the complete set of commuting Jucys–Murphy operators

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ABSTRACT

A method for evaluation of Kostka matrices at the level of bases, and determination of related irreducible basis of the Weyl duality is proposed. The method bases on Jucys–Murphy operators which constitute a complete set of commuting Hermitian operators along the general Dirac formalism of quantum mechanics, applied to the algebra of a symmetric group. The way of construction of appropriate projection operators is pointed out, and the combinatorial meaning of the path on the Young graph, corresponding to a standard Young tableau, is made transparent.

1. Introduction

The Heisenberg model of a magnet consisting of N spins s serves as a seminal example of the general scheme of the Schur–Weyl duality between the symmetric group Σ_N and the unitary group $U(n), n=2s+1$, both groups acting on the N-th tensor power space $\mathscr{H}=(\mathbb{C}^n)^{\otimes N}$, i.e. on the space of all quantum states of the magnet ([1,2]; cf. e.g. Refs. [3,4] for a recent compact presentation). Quantum computations within this scheme involve, in particular, an irreducible basis of the symmetric group Σ_N which we have proposed in some our earlier papers [5–7], referring to it as to Kostka matrices at the level of bases. Recently, essentially the same transformation matrices were introduced by Bacon et al. [8] under the name of the Schur transform, and applied within the context of quantum information processing. Their approach corresponds exactly to the above scheme of the Schur–Weyl duality, with the identification of the single-node space $\mathbb{C}^n$ as a qudit (a qubit for $n=2$), and of the whole magnet as a quantum register with the memory N, along standard definitions within quantum information theory [10]. These transformation matrices realise a (standard) irreducible basis in the carrier space of the permutation representation R^μ of Σ_N, acting (transitively) on the set of cosets of the Young subgroup specified by *the weight* μ, i.e. by a composition of N into not more than n nonnegative integers. The transitive representation R^μ decomposes as

$$R^\mu \cong \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu} \Delta^\lambda \tag{1}$$

into irreps of Σ_N, with the partition $\lambda \vdash N$ defining *the shape* of the corresponding irrep Δ^λ, $K_{\lambda\mu}$ are the famous Kostka numbers, and the sum runs over all partitions λ of N which are not smaller than μ in the dominance order. In this way, Kostka matrices at the level of bases are linear counterparts of Kostka numbers $K_{\lambda\mu}$ in the same way as Clebsch–Gordan coefficients (or appropriate Wigner 3j-symbols) are linear counterparts of the ordinary Clebsch–Gordan multiplicities $c(\lambda_1 \otimes \lambda_2, \lambda)$ for the inner product $\Delta^{\lambda_1} \otimes \Delta^{\lambda_2}$ of irreps in Σ_N.

A natural setting for derivation of explicit formulas for Kostka matrices at the level of bases is multiple coupling of N single-node

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