Uninorms and their applications

Paweł Drygaś

Uninorms and their applications

– Monograph –

WYDAWNICTWO UNIWERSYTETU RZESZOWSKIEGO
RZESZÓW 2023

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ISBN 978–83–8277–117–6

2046

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Wydanie I, format B5, ark. wyd. 8, ark. druk. 11,25, zlec. red. 74/2023

Druk i oprawa: Drukarnia Uniwersytetu Rzeszowskiego

To My Family

Contents

Preface

It does not matter how slowly you go as long as you do not stop.

Confucius

In human life, we constantly receive a lot of information from the world around us. We have to make decisions based on them, so we have to process and combine them. One way to combine information is to aggregate it (from a mathematical point of view). The data to be aggregated in each case may vary due to different types of information, from quantitative to qualitative information. Therefore, they should be processed in such a way that the data aggregated gave the expected results. These processes have been accompanying man for a very long time. The arithmetic mean was used as early as the time of the Babylonians.

However, a detailed study of aggregation functions in general and their formalization and classification is quite more recent, and has experienced a rapid growth from 1970's to nowadays.

Recently, many researchers have studied aggregation functions, and these operations have become an indispensable tool in many applications, from mathematics to the social sciences. For this reason, there is a lot of interest in aggregation functions and most of the known results on this topic have been summarized in several monographs entirely devoted to aggregation functions from both a theoretical and application point of view.

In addition, we can find tons of types of aggregation functions to choose from, depending on the area in which we want to use them. One of the possible divisions of aggregation is the division into conjunctive, disjunctive, average and mixed or hybrid aggregations. In this book we will take a look at uninorms, which are hybrid aggregations and show their main properties and some applications.

The key part of the monograph is the description of the original classification algorithms based on uninorms.

The described algorithms may be applied in decision support systems, for example, in medicine or other disciplines.

The book presents the main classes of uninorms presented by various authors. Some of the approaches have been supplemented and in some cases more intuitive approaches have been proposed.

In Chapter 1, some background information about aggregation, negation, or triangular norm and triangular conorm is provided, followed by basic information such as structure and some properties.

As it turns out, the only continuous uninorms are triangular norms and conorms. Therefore, various continuity weaknesses have been considered. The first one, leading to the class $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$, presents uninorms continuous on the boundary excluding the neutral element of the uninorm.

Another class of uninorms are representable uninorms. These are operations that are continuous beyond the two points $(0,1)$ and $(1,0)$. They can be represented in several ways: as isomorphic with addition in the set of real numbers, as isomorphic with multiplication in the set of nonnegativ real numbers, as strictly increasing.

The next step is to present the construction of ordinal sums in the Clifford sense [47]. This idea was used to construct t-norms and t-conorms, including the characterization of continuous t-norms and t-conorms. It was also used for the construction of uninorm (cf. [81, 82]), but a milestone in the application of this construction was the paper by Mesiarová-Zemánková, who resigned from ordering the indexes of intervals or subsets by the order corresponding to their elements. This approach allowed the characterization of several classes of uninorms and is also used to characterize many other classes of operations.

Next, uninorms continuous inside the unit square are shown. These uninorms can be represented as the ordinal sum of two continuous t-norms and a representable uninorm, or two continuous t-conorms and a representable uninorm.

Another type of uninorms are idempotent uninorms and locally internal uninorms on some subset of the unit square. They are described by a separating function that separates the minimum from the maximum. Moreover, the properties of the separating function have been described in such a way that it can be easily constructed.

All of the above considerations contributed in some way to the characterization of uninorms whose underlying t-norms and t-conorms are continuous. And so, in turn, we have a description of this class using points of discontinuity of a uninorm, set valued function, ordinal sum, and separating functions that separate the minimum and maximum from other values.

The next two sections present the general idea of describing two classes: uninorms that are locally internal to the boundary and uninorms that are not locally internal to the boundary. All possible values that the uninorm can take on the boundary are described in these sections and some conclusions are presented.

At the end of our theoretical considerations, the relationships between the individual classes were presented. As it turns out, they exhaust the set of all uninorms.

Part II consists of two threads. The first concerns the more theoretical application of uninorms to the construction of inverted fuzzy implications.

As shown in the papers [248, 249, 246], among the typical examples of fuzzy implications, there is the problem of inversibility over the entire unit square. Since the implication family forms a lattice, one way to solve the problem is to find the largest or smallest implication due to the inverse implication in a given subset (the partition depends on the selected family of implications) and combine them to obtain the optimal implication. In Chapter 4 we will show that the (UN)-implication is invertible with respect to the antecedent as well as the conclusion, assuming that the uninorm is representable, and that the family of (UN)-implications form an ordered family with respect to the parameter e, with a fixed generator of a uninorm. Thus, we can more easily choose the appropriate implication for the problem under consideration.

The second thread concerns the application of uninorms to the construction of multiclassifiers.

And so on in Chapter 6 we want to present the concept of uncertainty area of classifiers and an algorithm that uses uninorms to minimize the area of uncertainty in the prediction of new objects by complex classifiers (cf. [84]).

As it is easy to see, for the close neighborhood of the threshold parameter t very small differences in the classification weight can lead to opposing decisions. In addition, for objects whose classification weight was in the neighborhood of the threshold parameter, we make the most often classification errors. In order to avoid the incorrect classification, we propose to introduce an uncertainty area, which if the classifier returns the classification weight from the certain neighborhood of a threshold, will lead to abstain from the decision.

Furthermore, when classifying objects, we can construct different classifiers. We suggest aggregation of values obtained by the individual classifiers using uninorms. As a result, we build a new compound classifier, which additionally reduces the measure of uncertainty area, and thus increases the coverage (i.e. the number of classified objects) of the entire test data.

Another way to improve the quality of the multiclassifier is the method for selecting classifiers to build a multiclassifier. When classifying objects, we can construct different classifiers. Sometimes we get many classifiers that classify an object based on various premises (attributes) or different sources. Often the decisions obtained differ for some elements. Therefore a new classifier is being built that takes into account the weight of individual classifiers. Many of them are of low quality. Among others things, for this reason, in general, it gives better results than individual classifiers. The use of all classifiers (especially those of low quality) does not always give satisfactory results. However, the use of all classifiers and their later aggregation is sometimes very expensive. Therefore, we present a method that allows to eliminate some classifiers while increasing the quality of classification. Our approach is characterized by, compared to the majority of existing ones, that classifiers are not only aggregated, but dynamically selected when testing a particular test object. The purpose of this selection is to improve the global quality of the classification and it is based on the raw results of the test object classification by all aggregated classifiers or based on certain selection parameters learned from training data. This approach is based on the paper [23].

In Chapter 8, we will create models to detect phishing e-mails. Prediction of phishing e-mails can be based on two sources of information: e-mail content and links in e-mails. Although the links appear in the body of the e-mail, both of these sources of information are usually treated separately and require the development of special methods of their analysis. So, the analysis of datasets containing e-mails content and datasets containing links from e-mails results in two classifiers. The first one is able to classify the content of an e-mail by predicting phishing content. The second one allows to predict whether any link in the message is a phishing link. If, when classifying a specific e-mail, both of the above classifiers match, the situation is clear. However, if they do not match, then we need to propose a method for resolving the conflict between these classifiers. This method can be based on various approaches. In this chapter, we propose a method based on the aggregation of the classification weights of both classifiers, using the methods of the fuzzy set theory. Both of the above-mentioned classifiers, together with the established method of resolving conflicts between them, can be treated as a complex classifier for recognizing phishing. Therefore, the use of an optimally selected method of resolving the above-mentioned conflict is of key importance for the effectiveness of the complex classifier.

The presented results may be useful not only for the community working on fuzzy sets and their extensions, but also for researches and practitioners dealing with the problems of classifiers and Petri Nets. It can serve as a brief introduction into the theory of uninorms as aggregation functions and application in classification problems. It can also be used as supplementary reading for the students of mathematics and computer science.

I would like to thank Professors that helped me in better understanding the essence and nuances of fuzzy sets theory, its extensions, and applications. Namely, these are the following persons (listed in the alphabetical order): Michał Baczyński, Jan G. Bazan, Humberto Bustince, Józef Drewniak, Janusz Kacprzyk, Martin Kalina, Radko Mesiar and Zbigniew Suraj. I am also grateful to my colleagues from Poland and abroad with whom I cooperated working on scientific problems. Especially, I would like to thank my colleagues from the University of Rzeszów, notably Urszula Bentkowska, Anna Król, Zofia Matusiewicz, Barbara Pękala, Ewa Rak, with whom we discussed scientific problems for many hours at seminars. Finally, I would like to express my deepest gratitude to my family and friends.

Rzeszów, *Paweł Drygaś*
August 2023

Part I
Theoretical aspects of uninorms

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established.

L.A. Zadeh

Uninorms are a special kind of aggregation functions that generalise both t-norms and t-conorms. They appeared for the first time in [107] and [268] (although the very related operators called Dombi's operators were already studied in [70], compensatory operators considered in [141] and idempotent operations were considered by Czogała and Drewniak in [52]) with the idea of allowing certain kind of aggregation operators combining the maximum and the minimum, depending on an element $e \in [0,1]$.

The idea of uninorms was deeper studied in [102], where the structure of such operators was analysed and two first classes of uninorms were introduced: uninorms in $\mathscr{U}_{\min}$ (given by minimum on $A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e)$), and $\mathscr{U}_{\max}$ (given by maximum on $A(e)$), and representable uninorms $\mathscr{U}_{rep}$ (extremely related with Dombi's operators introduced in [70], see also [71]). After this, some other classes of uninorms were introduced and characterized. In this part, we would like to present the known classes of uninorms and their characterizations, referring to the literature wherever possible and supplementing them with new characterizations of the remaining classes.

So we start from a more intuitive case - the unit interval, to end up on an arbitrary bounded lattice.

Chapter 1
Uninorms on the unit interval

Mathematics is written for mathematicians.

N. Copernicus

Uninorms are a hybrid operations build with the use of t-norms, t-conorms, and means. Therefore, we will first discuss the operations mentioned above and other functions used later, and then we will proceed to the characterization of certain classes of uninorms.

1.1 Algebraic properties of operations

In this part, we will present some selected properties of operations that will be used later in the book.

Definition 1.1 ([103, 143]). The operation $F : [0,1]^2 \to [0,1]$ is:

associative $\Leftrightarrow \underset{x,y,z\in[0,1]}{\forall} \quad F(x,F(y,z)) = F(F(x,y),z)$ (1.1)

idempotent $\Leftrightarrow \underset{x\in[0,1]}{\forall} \quad F(x,x) = x$ (1.2)

increasing $\Leftrightarrow \underset{x,y,z\in[0,1]}{\forall} \quad (x \le y) \Rightarrow (F(x,z) \le F(y,z), F(z,x) \le F(z,y))$ (1.3)

stict increasing $\Leftrightarrow \underset{x,y,z\in[0,1]}{\forall} \quad (x < y) \Rightarrow (F(x,z) < F(y,z),\ F(z,x) < F(z,y))$ (1.4)

commutative $\Leftrightarrow \underset{x,y\in[0,1]}{\forall} \quad F(x,y) = F(y,x)$ (1.5)

We say that operation F has:

$$\textit{idempotent element} \quad \Leftrightarrow \quad \underset{s\in[0,1]}{\exists} \quad F(s,s)=s, \tag{1.6}$$

$$\textit{neutral element} \quad \Leftrightarrow \quad \underset{e\in[0,1]}{\exists} \ \underset{x\in[0,1]}{\forall} \quad F(e,x)=F(x,e)=x, \tag{1.7}$$

$$\textit{zero element} \quad \Leftrightarrow \quad \underset{z\in[0,1]}{\exists} \ \underset{x\in[0,1]}{\forall} \quad F(z,x)=F(x,z)=z. \tag{1.8}$$

Definition 1.2 ([103, 143]). Let $F : [0,1]^2 \to [0,1]$ be binary operation.
The n-th power of $x \in [0,1]$ is defined as follows:

$$x_F^n = \begin{cases} x & \text{if } n = 1, \\ F(x_F^{n-1}, x) & \text{otherwise.} \end{cases}$$

Definition 1.3 (cf. [103]). Let $F : [0,1]^2 \to [0,1]$ be increasing associative binary operation with neutral element $e \in [0,1]$. We say that operation F fulfill Archimedean property if:

$$\underset{x,y\in(0,1)}{\forall} \left(x,y < e \Rightarrow \underset{n\in\mathbb{N}}{\exists} \ x^n < y \right),$$
$$\underset{x,y\in(0,1)}{\forall} \left(x,y > e \Rightarrow \underset{n\in\mathbb{N}}{\exists} \ x^n > y \right).$$

Definition 1.4 ([103, 143]). Let $F : [0,1]^2 \to [0,1]$ be associative, increasing binary operation with zero element $z \in [0,1]$.

- An element $x \in (0,1)$ is called a nilpotent element of F if there exists some $n \in \mathbb{N}$ such that $x_F^n = 0$.
- An element $x \in (0,1)$ is called a zero divisor of F if there exists some $y \in (0,1)$ such that $F(x,y) = z$.
- F is called a nilpotent operation if all elements are nilpotent.
- F is called a positive operation if it has no zero divisors.

Definition 1.5 ([103]). Let (X,F) be a lattice ordered semigroup.
(X,F) is called negatively ordered semigroup if $F(x,y) \le x \wedge y$ for all $x,y \in X$.
(X,F) is called positively ordered semigroup if $F(x,y) \ge x \vee y$ for all $x,y \in X$.

1.2 Aggregation operators

Firstly, we recall definition of an aggregation function. More details can be found in [37, 114, 214, 27, 36]

Definition 1.6 (cf. [35]). A function $A : [0,1]^n \to [0,1]$, $n \in \mathbb{N}$, $n \ge 2$, which is increasing in each variable, i.e.

$$(\forall_{1\le i\le n} s_i \le t_i) \Rightarrow A(s_1,\dots,s_n) \le A(t_1,\dots,t_n), \tag{1.9}$$

for all $s_1,\ldots,s_n,t_1,\ldots,t_n \in [0,1]$ is called an aggregation function (aggregation operator) if $A(0,\ldots,0)=0, A(1,\ldots,1)=1$.

Definition 1.7 ([35]). Let $n \geq 2$. $A:\mathbb{R}^n \to \mathbb{R}$ is a mean (average function) if it is increasing and idempotent, i.e.

$$\forall_{s,t\in\mathbb{R}^n}\,(\forall_{1\leq k\leq n}s_k \leq t_k) \Rightarrow A(s_1,...,s_n) \leq A(t_1,...,t_n),$$

and

$$\forall_{t\in\mathbb{R}}A(t,...,t)=t.$$

Lemma 1.1. *For every mean A we have*

$$\forall t \in \mathbb{R}^n \min_{1\leq k\leq n} t_k \leq A(t_1,...,t_n) \leq \max_{1\leq k\leq n} t_k. \tag{1.10}$$

From the above lemma we see that the mean can be restricted to any interval. Our domain of interest is the interval $[0,1]$. In this case, the mean is the aggregation function.

Example 1.1. Let $n=2$. The basic aggregation operations are:

- arithmetic mean
$$A(x,y)=\frac{x+y}{2}, \quad x,y\in\mathbb{R},$$
- geometric mean
$$G(x,y)=\sqrt{xy}, \quad x,y\in\mathbb{R}_+,$$
- harmonic mean
$$H(x,y)=\frac{2}{\frac{1}{x}+\frac{1}{y}}, \quad x,y\in\mathbb{R}_+,$$
- projections
$$P_1(x,y)=x, \quad x,y\in\mathbb{R},$$
$$P_2(x,y)=y, \quad x,y\in\mathbb{R}.$$

Example 1.2. Let $\varphi:[0,1]\to[0,1]$ be an increasing bijection and $t,w\in[0,1]^n$. We remind here two important examples of aggregation function:

- the quasi-arithmetic mean (cf. [2])
$$A(t_1,...,t_n)=\varphi^{-1}(\frac{1}{n}\sum_{k=1}^{n}\varphi(t_k)),$$
- the generalized weighted average (cf. [35])
$$A(t_1,...,t_n)=\varphi^{-1}(\sum_{k=1}^{n}w_k\varphi(t_k)).$$

where $\sum_{k=1}^{n} w_k = 1$.

- the median function

$$med(t_1,...,t_n) = \begin{cases} y_{(n+1)/2} & \text{if } n \text{ is odd,} \\ \frac{y_{n/2}+y_{(n/2)+1}}{2} & \text{if } n \text{ is even.} \end{cases}$$

  where $(y_1,...,y_n)$, $y_1 \geq y_2 \geq ... \geq y_n$ is a sequence of arguments $(t_1,...,t_n)$ ordered decreasingly.

1.3 Fuzzy negations

Now, let us recall the notion of a fuzzy negation

Definition 1.8 ([17], p. 13). A function $N : [0,1] \to [0,1]$ is called a fuzzy negation if it is decreasing and

$$N(0) = 1, \quad N(1) = 0. \tag{1.11}$$

A fuzzy negation N is called strict if, in addition, N is strictly decreasing and continuous.
A fuzzy negation N is called strong if it is an involution, i.e.,

$$N(N(x)) = x, \ \ x \in [0,1].$$

Example 1.3 ([17], p. 14–15). The least and the greatest fuzzy negations are of the form

$$N_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}, \qquad N_1(x) = \begin{cases} 1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \end{cases}.$$

Continuous negations

$$N_K(x) = 1 - x^2, \quad x \in [0,1],$$
$$N_R(x) = 1 - \sqrt{x}, \quad x \in [0,1].$$

Standard (classical) negation proposed by Zadeh in 1965 is the function

$$N_S(x) = 1 - x \ \text{ for } \ x \in [0,1].$$

Sugeno class of negation

$$N^{\lambda}(x) = \frac{1-x}{1+\lambda x}, \quad \lambda \in (-1,\infty).$$

Yager class of negation

$$N^{w}(x) = (1-x^{w})^{\frac{1}{w}}, \quad w \in (0,\infty).$$

Theorem 1.1 ([17]). *If a function $N : [0,1] \to [0,1]$ is decreasing and involutive, then it also satisfies* (1.11) *and it is continuous. Moreover, N is a bijection.*

Corollary 1.1. *Every strong negation is strict.*

Theorem 1.2 ([17]). *Every continuous fuzzy negation N has the unique fixed point, i.e., there exists $e \in (0,1)$ such that $N(e) = e$.*

Theorem 1.3 ([17]). *If $\varphi : [0,1] \to [0,1]$ is increasing bijection and N is a fuzzy (strict, strong) negation, then N_φ (where $N_\varphi(x) = \varphi^{-1}(N(\varphi(x))))$ is also a fuzzy (strict, strong) negation.*

Theorem 1.4 ([17]). *For a function $N : [0,1] \to [0,1]$ the following statements are equivalent:*

(i) N is a strong negation.
(ii) There exists increasing bijection φ such that

$$N(x) = \varphi^{-1}(1 - \varphi(x)), x \in [0,1].$$

(iii) There exists a strictly increasing, continuous function $g : [0,1] \to [0,\infty)$ such that $g(0) = 0$ and

$$N(x) = g^{-1}(g(1) - g(x)), x \in [0,1].$$

(iv) There exists a strictly decreasing, continuous function $f : [0,1] \to [0,\infty)$ such that $f(1) = 0$ and

$$N(x) = f^{-1}(f(0) - f(x)), x \in [0,1].$$

Example 1.4. The negations N_K and N_R from the Example 1.3 are strict negations, while the negations N_S, N^λ and N^w are additionally strong negations.
Let $a \in (0,1)$. Operation $N_{\underline{a}}$ given by

$$N_{\underline{a}}(x) = \begin{cases} (1 - \frac{1}{a})x + 1 & if\, x \leq a \\ \frac{a}{a-1}x + \frac{a}{1-a} & otherwise \end{cases}$$

is strong negation (see Figure 1.1).

1.4 Triangular norms

Triangular norms were originally introduced by Menger [194], when generalizing the triangle inequality from the classical metric spaces to the probabilistic metric spaces. Schweizer and Sklar [237] redefined axioms of triangular norms into the form used today. From the viewpoint of fuzzy logic, the triangular norms are suitable candidates for the generalization of the classical binary conjunction into a fuzzy intersection (see Klir and Yuan [145] or Gottwald [113]). The following definitions and results, with proofs, can be found in the monograph by Klement et al. [143]. First we start with basic definitions and some properties of triangular norms.

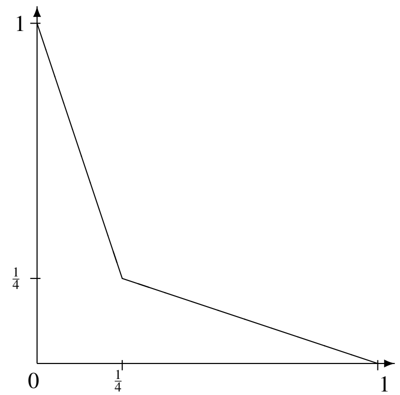

Fig. 1.1 Negation $N_{\underline{1/4}}$ from Example 1.4

Definition 1.9 ([143]). Operation $T : [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element $e = 1$.

Definition 1.10 ([129]). Operation T is called a triangular subnorm if it is commutative, associative, increasing with respect to both variables and fulfils the condition $T \leq \min$.

Example 1.5 ([143]). These are the basic triangular norms:

$T_M(x,y) = \min(x,y), \quad x,y \in [0,1],$ (minimum)

$T_P(x,y) = x \cdot y, \quad x,y \in [0,1],$ (product)

$T_L(x,y) = \max(x+y-1,0), \quad x,y \in [0,1],$ (Łukasiewicz)

$$T_H(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ \frac{xy}{x+y-xy} & \text{otherwise,} \end{cases} \quad \text{(Hamacher)}$$

$$T_E(x,y) = \frac{xy}{2-x-y+xy}, \quad x,y \in [0,1], \quad \text{(Einstein)}$$

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{(drastic)}$$

$$T_{nM}(x,y) = \begin{cases} 0 & \text{if } x+y \leq 1, \\ \min(x,y) & \text{otherwise.} \end{cases} \quad \text{(nilpotent-minimum)}$$

Certainly every triangular norms are triangular subnorms. Operations
$T_1(x,y) = 0, \quad x,y \in [0,1],$
$T_2(x,y) = \frac{1}{2}x \cdot y, \quad x,y \in [0,1]$
are triangular subnorms, but not triangular norms.

Remark 1.1. We have the following order between basic t-norms:

$$T_D \leq T_L \leq T_E \leq T_P \leq T_H \leq T_M.$$

These operations may be used to construct new triangular norms.

Lemma 1.2. *A continuous t-norm is an Archimedean t-norm if* $T(x,x) < x$ *for all* $x \in (0,1)$.

Example 1.6. T_P, T_H, T_E are strictly increasing, continuous and Archimedean t-norms.
T_L is continuous, nilpotent and Archimedean t-norm.
T_M is the only idempotent t-norm.

Archimedean t-norms have representations using additive generators as well as using isomorphism with basic t-norms.

Theorem 1.5 (cf. [143]). *Operation* T *is a continuous Archimedean t-norm if and only if it is isomorphic either to* T_P *or to* T_L.

Theorem 1.6 (cf. [143]). *Operation* T *is a continuous Archimedean t-norm if and only if* T *has a continuous additive generator, i.e. , there exists a continuous, strictly decreasing function* $t : [0,1] \to [0,\infty]$ *with* $t(1) = 0$*, which is uniquely determined up to a positive multiplicative constant, such that for all* $(x,y) \in [0,1]^2$ *we have* $T(x,y) = t^{(-1)}(t(x)+t(y))$. *The function* t *is called the additive generator of t-norm* T.

In above theorem $f^{(-1)}$ denotes the pseudo-inverse of f. In this case we can assume that $f^{(-1)}(x) = f^{-1}(\min(f(0),x))$. For more details the reader can see [143].

Using the above descriptions we can characterize all continuous t-norms. For this, however, we need an ordinal sum construction, which we will provide in Section 1.8. There we will also present the characterization of all continuous t-norms (see Theorem 1.19).

1.5 Triangular conorms

Definition 1.11 ([143]). Operation $S : [0,1]^2 \to [0,1]$ is called a triangular conorm (t-conorm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element $e = 0$.

Definition 1.12 ([129]). Operation S is called a triangular superconorm if it is commutative, associative, increasing with respect to both variables and fulfils the condition $S \geq \max$.

Theorem 1.7 ([143]). *A function* $S : [0,1]^2 \to [0,1]$ *is a t-conorm if and only if there exists a t-norm* T *such that for all* $(x,y) \in [0,1]^2$ *we have* $S(x,y) = 1 - T(1-x, 1-y)$.

Example 1.7 ([143]). These are the basic triangular conorms:

$S_M(x,y) = \max(x,y), \quad x,y \in [0,1],$ (maximum)

$S_P(x,y) = x+y-xy, \quad x,y \in [0,1],$ (probabilistic sum)

$S_L(x,y) = \max(x+y,1), \quad x,y \in [0,1],$ (Łukasiewicz triangular norm)

$$S_H(x,y) = \begin{cases} 1 & \text{if } x=y=1, \\ \frac{x+y-2xy}{1-xy} & \text{otherwise,} \end{cases}$$ (Hamacher)

$S_E(x,y) = \frac{x+y}{1+xy}, \quad x,y \in [0,1],$ (Einstein)

$$S_D(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y)=0, \\ 1 & \text{otherwise.} \end{cases}$$ (drastic)

Remark 1.2. We have the following order between basic t-conorms:

$$S_M \leq S_H \leq S_P \leq S_E \leq S_L \leq S_D.$$

Lemma 1.3. *A continuous t-conorm is an Archimedean t-norm if $S(x,x) > x$ for all $x \in (0,1)$.*

Example 1.8. S_P, S_H, S_E are strictly increasing, continuous, Archimedean t-conorms.
S_L is continuous, nilpotent and Archimedean t-conorm.
S_M is the only idempotent t-conorm.

Archimedean t-conorms have representations using additive generators as well as using isomorphism with the basic t-conorms.

Theorem 1.8 (cf. [143]). *Operation S is a continuous Archimedean t-conorm if and only if it is isomorphic either to S_P or to S_L.*

Theorem 1.9 (cf. [143]). *Operation S is a continuous Archimedean t-conorm if and only if S has a continuous additive generator, i.e. , there exists a continuous, strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that for all $(x,y) \in [0,1]^2$ we have $S(x,y) = s^{(-1)}(s(x)+s(y))$. The function s is called the additive generator of t-conorm S.*

In the above theorem $f^{(-1)}$ denotes the pseudo-inverse of f. In this case we can assume that $f^{(-1)}(x) = f^{-1}(\min(f(1),x))$. For more details the reader can see [143].

Using the above descriptions we can characterize all continuous t-conorms. For this, however, we need an ordinal sum construction, which we will provide in Section 1.8. There we will also present the characterization of all continuous t-conorms (see Theorem 1.21).

1.6 Notion of uninorms

Let us start with the definition of uninorms on the unit interval.

Definition 1.13 ([268]). Operation $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is commutative, associative, increasing and has the neutral element $e \in [0,1]$. The set of all uninorm will be denoted by $\mathscr{U}$.

Remark 1.3 (cf. [268]). If a uninorm U has the neutral element $e = 1$ then it is a triangular norm.
If a uninorm U has the neutral element $e = 1$ then it is a triangular conorm.

In the case $e \in (0,1)$ we say that uninorm U is a proper uninorm composed by using a triangular norm and a triangular conorm.

Theorem 1.10 ([268, 102]). *Let U be a uninorm with neutral element $e \in (0,1)$. Then there exists a t-norm T and a t-conorm S such that U on $[0,e]^2 \cup [e,1]^2$ is given by*

$$U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2. \end{cases} \tag{1.12}$$

We will often denote the underlying t-norm used in the construction of the uninorm by T_U and the t-conorm by S_U to emphasize their relationship to the uninorm. Moreover, when the underlying t-norm and t-conorm are known, we will denote the uninorm by $U_{T,S,e}$.

Lemma 1.4 (cf. [102]). *If operation $U : [0,1]^2 \to [0,1]$ is increasing and has the neutral element $e \in (0,1)$, then*

$$\min \le U \le \max \ \text{in } A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e).$$

Furthermore, if U is associative, then $U(0,1), U(1,0) \in \{0,1\}$.

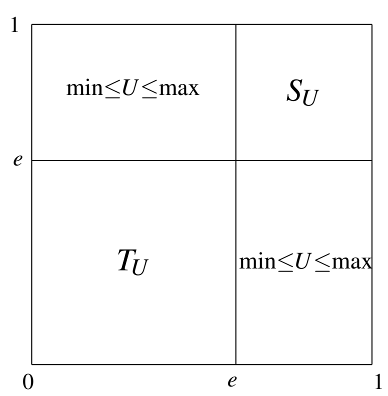

Fig. 1.2 The structure of uninorm

If U is a uninorm, then in the first case ($U(0,1) = 0$) we call it a conjunctive uninorm and if it gives the appropriate form of uninorm we will denote by U^c, and in the second case ($U(0,1) = 1$) a disjunctive uninorm and if it gives the appropriate form of uninorm we will denote by U^d.

Example 1.9. Using the orders for t-norms and t-conorms (see Remark 1.1 and 1.2) and boundary values on $A(e)$ we can define the following uninorms

$$\underline{U_e}(x,y) = \begin{cases} 0 & \text{if } x,y \in [0,e), \\ \max(x,y) & \text{if } x,y \in [e,1], \\ \min(x,y) & \text{elsewhere}, \end{cases}$$

$$\overline{U_e}(x,y) = \begin{cases} \min(x,y) & \text{if } x,y \in [0,e], \\ 1 & \text{if } x,y \in (e,1], \\ \max(x,y) & \text{elsewhere}. \end{cases}$$

Moreover,

$$\underline{U_e} \leq U \leq \overline{U_e}$$

for arbitrary uninorm U. Hence, the uninorm $\underline{U_e}$ is called the least uninorm and the uninorm $\overline{U_e}$ is called the greatest uninorm.

Theorem 1.11 (cf. [52]). *If a uninorm is continuous then $e = 0$ or $e = 1$.*

Theorem 1.12. *Let U be a uninorm with the neutral element $e \in (0,1)$. If the underlying t-norm is continuous in $(1,1)$ and the underlying t-conorm is continuous in $(0,0)$ then uninorm U is continuous in (e,e).*

Proof. Since T_U is continuous in $(1,1)$ and S_U is continuous in $(0,0)$, and since U is commutative, we only have to check that for two monotone sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = e = \lim_{n\to\infty} b_n$ and $a_n < e$, $b_n > e$ for $n \in \mathbb{N}$ we have $\lim_{n\to\infty} U(a_n,b_n) = e$. However, monotonicity gives us the following $a_n = U(a_n,e) \leq U(a_n,b_n) \leq U(e,b_n) = b_n$ and thus $e \leq \lim_{n\to\infty} U(a_n,b_n) \leq e$. □

Recall that there are no continuous uninorms with neutral element $e \in (0,1)$, and then continuity in some subset of $[0,1]^2$ has been considered in order to define some classes of uninorms. Below, we will present the previously mentioned classes of uninorms.

1.7 Uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$

According to the Lemma 1.4, the extreme values on $A(e)$ are min and max. The following theorem determines when this is the only possibility.

Theorem 1.13 ([268]). *Let U be a uninorm with neutral element $e \in (0,1)$ and both functions $f_1(x) = U(x,1)$ and $f_0(x) = U(x,0)$ ($x \in [0,1]$) are continuous except perhaps at the point $x = e$. Then U is given by one of the following forms.*

(i) If $U(0,1) = 0$*, then*

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & if\ (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & if\ (x,y) \in [e,1]^2, \\ \min(x,y) & otherwise. \end{cases} \tag{1.13}$$

(ii) If $U(0,1) = 1$*, then*

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & if\ (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & if\ (x,y) \in [e,1]^2, \\ \max(x,y) & otherwise. \end{cases} \tag{1.14}$$

Denote $\mathscr{U}_{\min}$ the class of uninorms having form (1.13) and $\mathscr{U}_{\max}$ the class of uninorms with form (1.14).

Example 1.10. Let U be a uninorm such that T and S are Łukasiewicz t-norm and t-conorm. Then U is given by one of the following form:

i) $U^{\min}_{L,L,e}(x,y) = \begin{cases} \max(0, \min(x+y-e, 1)) & \text{if } (x,y) \in [0,e]^2 \cup [e,1]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$

ii) $U^{\max}_{L,L,e}(x,y) = \begin{cases} \max(0, \min(x+y-e, 1)) & \text{if } (x,y) \in [0,e]^2 \cup [e,1]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$

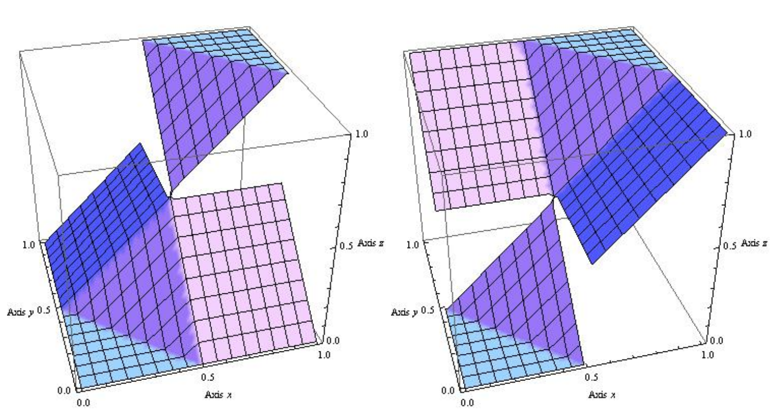

Fig. 1.3 Uninorms with $e = 0.5$ and Łukasiewicz underlying t-norm and t-conorm from class $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$

1.8 Ordinal sum of uninorms

At first the theorem about the ordinal sum was given for semigroups which has the same kind of order (see Climescu [48], Clifford [47] and Jenei [130]).

Theorem 1.14 ([47]). *Let (X,F), (Y,G) be semigroups, $X \cap Y = \emptyset$ and H be given by*

$$H(x,y) = \begin{cases} F(x,y) & \text{if } x,y \in X, \\ G(x,y) & \text{if } x,y \in Y, \\ x & \text{if } x \in X,\ y \in Y, \\ y & \text{if } x \in Y,\ y \in X. \end{cases} \tag{1.15}$$

Then $(X \cup Y, H)$ is a semigroup and it is called an ordinal sum of semigroups (X,F) and (Y,G).

Let K be given by

$$K(x,y) = \begin{cases} F(x,y) & \text{if } x,y \in X, \\ G(x,y) & \text{if } x,y \in Y, \\ y & \text{if } x \in X,\ y \in Y, \\ x & \text{if } x \in Y,\ y \in X. \end{cases} \tag{1.16}$$

Then $(X \cup Y, K)$ is a semigroup and it is called a dual ordinal sum of semigroups (X,F) and (Y,G).

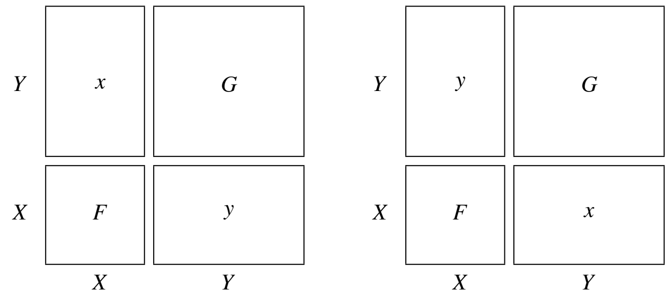

Fig. 1.4 Sums (1.15) of semigroups (X,F), (Y,G) and its dual version given by (1.16)

Theorem 1.15 ([47]). *Let $A \neq \emptyset$ be a totally ordered set and $\{G_t\}_{t \in A}$ with $G_t = (X_t, *_t)$ be a family of semigroups. Assume that for all $t,s \in A$ with $t < s$ the sets X_t and X_s are either disjoint or that $X_t \cap X_s = \{x_{t,s}\}$, where $x_{t,s}$ is both the neutral element of G_t and the annihilator of G_s and where for each $r \in A$ with $t < r < s$ we have $X_r = \{x_{t,s}\}$. Put $X = \bigcup_{t \in A} X_t$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_t y & if\ (x,y) \in X_t \times X_t, \\ x & if\ (x,y) \in X_t \times X_s \text{ and } t < s, \\ y & if\ (x,y) \in X_t \times X_s \text{ and } t > s. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $t \in A$ the semigroup G_t is commutative.

Theorem 1.16 ([47]). *Let $(X, F; \leq)$, $(Y, G; \leq)$ be disjoint linearly ordered semigroups, and H be given by*

$$H(x,y) = \begin{cases} F(x,y) & if\ x,y \in X, \\ G(x,y) & if\ x,y \in Y, \\ x & if\ x \in X,\ y \in Y, \\ y & if\ x \in Y,\ y \in X. \end{cases}$$

The semigroup $(X \cup Y, H)$ is negatively ordered, if and only if semigroups (X, F), (Y, G) are negatively ordered.

This theorem has been applied to the construction of the ordinal sum of t-norm (see Theorem 1.18). A different approach can be found in [73], where uninorm constructions were considered, first noting that uninorms of the family $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ (see Theorem 1.13) are the ordinal sums of t-norm and t-conorm.

Theorem 1.17 ([73]). *Let $(X, F, \leq)$, $(Y, G, \leq)$ be ordered semigroups and H be given by*

$$H(x,y) = \begin{cases} F(x,y) & if\ x,y \in X, \\ G(x,y) & if\ x,y \in Y, \\ x & if\ x \in X,\ y \in Y, \\ y & if\ x \in Y,\ y \in X. \end{cases}$$

$(X \cup Y, H)$ is an ordered semigroup if and only if $(X, F, \leq)$ is negatively ordered.

1.8.1 Construction of t-norms

Theorem 1.18 (cf. [130]). *Let $\{[a_k, b_k]\}_{k \in \mathscr{I}}$ be a countable family of nonoverlapping, closed, proper subintervals of $[0,1]$. Let T be an operation in $[0,1]$ defined by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k} \right) & if\ x,y \in (a_k, b_k], \\ \min(x,y) & otherwise, \end{cases} \tag{1.17}$$

where T_k are triangular subnorms. Moreover, we assume that the operation T_k have neutral element $e = 1$, if $b_k = a_l$ and T_l is with zero divisors, or $b_k = 1$. Then T is a triangular norm.

Operation given by (1.17) is called the ordinal sum of $\{([a_k, b_k], T_k)\}_{k \in \mathscr{T}}$ and each T_k is called a summand.

Example 1.11. Operation

$$T(x,y) = \begin{cases} 0 & if \ \ x,y \in (0,\frac{1}{2}], \\ 2xy - x - y + 1 & if \ \ x,y \in (\frac{1}{2},1], \\ \min(x,y) & otherwise, \end{cases}$$

is an ordinal sum of operations $T_1 = 0$ and $T_2 = T_P$ where $a_1 = 0$, $b_1 = a_2 = \frac{1}{2}$, $b_2 = 1$.

In this example we can see, that operation T_2 has no zero divisors and interval $(\frac{1}{2},1]$ is closed with respect to operation T, but interval $[\frac{1}{2},1]$ is not closed with respect to operation T.
Operation T_1 has zero divisors and $[0,\frac{1}{2}]$ is closed with respect to operation T.

Example 1.12. Operation

$$T(x,y) = \begin{cases} 0 & if \ \ x,y \in (0,\frac{1}{2}], \\ \max(x+y-1,\frac{1}{2}) & if \ \ x,y \in (\frac{1}{2},1], \\ \min(x,y) & otherwise, \end{cases}$$

is built by using (1.17), where $T_1 = 0$ and $T_2 = T_L$, but it is non-associative, because T_1 has no neutral element and T_L has zero divisors, e.g., for $x = \frac{1}{2}$, $y = z = \frac{3}{4}$, $T(\frac{1}{2}, T(\frac{3}{4},\frac{3}{4})) = T(\frac{1}{2},\frac{1}{2}) = 0$, $T(T(\frac{1}{2},\frac{3}{4}),\frac{3}{4}) = T(\frac{1}{2},\frac{3}{4}) = \frac{1}{2}$. Therefore $T(T(x,y),z) \neq T(x,T(y,z))$.

Using the above construction we can characterize all continuous t-norms.

Theorem 1.19 ([143], Theorem 5.11). *For an operation* $T : [0,1]^2 \to [0,1]$ *the following items are equivalent:*

(i) T *is a continuous t-norm.*
(ii) T *is uniquely representable as an ordinal sum of continuous Archimedean t-norms.*

Remark 1.4. Note that in the construction (1.17) we obtain the t-norm T_M when $\mathscr{T}$ is the empty set.

1.8.2 Construction of t-conorms

Theorem 1.20 (cf. [130]). *Let* $\{[a_k, b_k]\}_{k \in \mathscr{S}}$ *be a countable family of nonoverlapping, closed, proper subintervals of* $[0,1]$. *Let* S *be an operation in* $[0,1]$ *defined by*

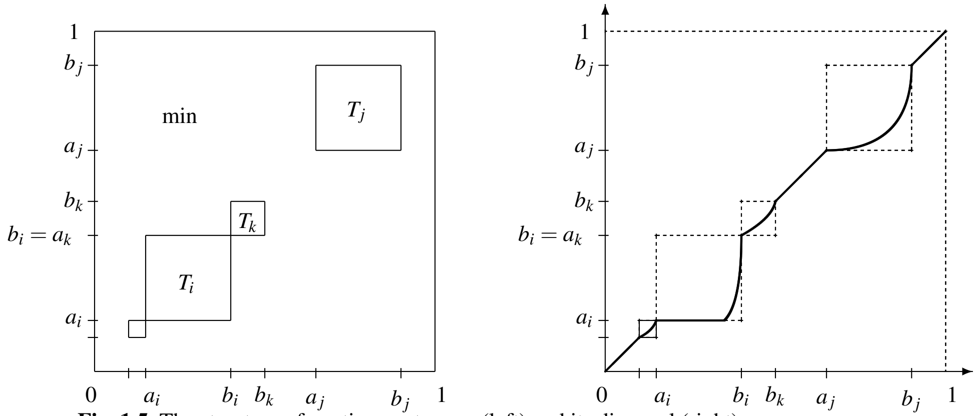

Fig. 1.5 The structure of continuous t-norm (left) and its diagonal (right)

$$S(x,y) = \begin{cases} a_k + (b_k - a_k) S_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & if \;\; x,y \in [a_k, b_k), \\ \max(x,y) & otherwise, \end{cases} \qquad (1.18)$$

where S_k are triangular superconorms. Moreover, we assume that the operation S_k have neutral element $e = 0$, if $a_k = b_l$ and S_l is with zero divisors, or $a_k = 0$. Then S is a triangular conorm.

Operation given by (1.18) is called the ordinal sum of $\{([a_k, b_k], S_k)\}_{k \in \mathscr{S}}$ and each S_k is called a summand.

Using the above construction we can characterize all continuous t-conorms.

Theorem 1.21 ([143]). *For an operation $S : [0,1]^2 \to [0,1]$ the following items are equivalent:*

(i) S is a continuous t-conorm.
(ii) S is uniquely representable as an ordinal sum of continuous Archimedean t-conorms.

Remark 1.5. Note that in the construction (1.18) we obtain the t-conorm S_M when $\mathscr{S}$ is the empty set.

1.8.3 Construction of uninorms

Theorem 1.22 (cf. [73]). *Let $([0,s], F)$ be a negatively ordered, commutative semigroup with the neutral element s, $([s,1], G)$ be an ordered, commutative semigroup with the neutral element $e \in [s,1]$ and $G(s,s) = s$. Then the ordinal sum of F and G is a uninorm with the neutral element e.*

Another application of these results can be found in [78] and [81], which can be described as an algorithm for constructing idempotent uninorms.

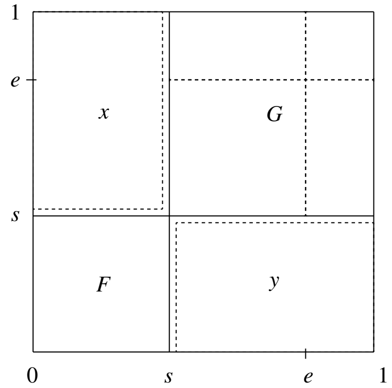

Fig. 1.6 The uninorm from Theorem 1.22

Let us take an interval $[a,b] \subset [0,1]$ and by isomorphism transform the semigroup with the arbitrary order from $[0,1]$ to $[a,b]$ (here we may take a uninorm). Then, let us add a semigroup in the left or right side of interval $[a,b]$ in the following way:

- If we add an interval $[c,a)$ (in the left side), then we take the negatively ordered semigroup F and use the ordinal sum construction (see (1.15)).
- If we add the interval $(b,c]$ (in the right side), then we take the positively ordered semigroup G and use the dual ordinal sum construction (see (1.16)).
- In this construction we may take open or close or left open or right open intervals, but the set on which the operation is defined must be an interval or a point.
- We continue this construction until we obtain interval $[0,1]$.

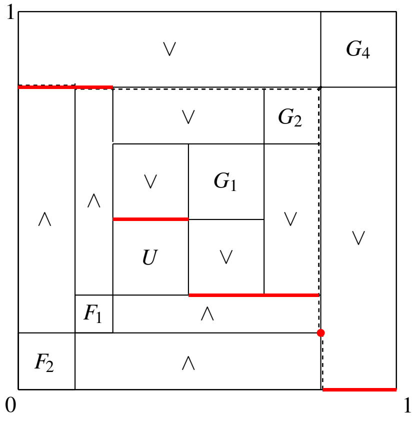

Fig. 1.7 Example of the uninorm constructed by using ordinal sum theorem

This type of ordinal sum has been considered in several papers and conference papers (see [73, 81]). Each time assuming that the order in the set of indices is consistent with the order in the set $[0,1]$. A different approach can be found in the works of A. Mesiarová-Zemánková. This approach allowed, for example, to use of representable uninorms as a summand and it is one giant leap for uninorms description. First, we will present the basic notations that allow us to use the Theorem 1.15 with regard to uninorms, as with t-norms (see Theorem 1.18)

Let $e \in (0,1)$ and let K be an index set which is finite or countably infinite, $\{(a_k,b_k)\}_{k\in K}$ be a disjoint family of open subintervals (which can be also empty) of $[0,e]$, such that $\bigcup_{k\in K}[a_k,b_k] = [0,e]$, $\{(c_k,d_k)\}_{k\in K}$ be a disjoint family of open subintervals (which can be also empty) of $[e,1]$, such that $\bigcup_{k\in K}[c_k,d_k] = [e,1]$.

The isomorphism between uninorm U_k and the respective semigroup $U_{\nu_k}^{a_k,b_k,c_k,d_k}$ on $[a_k,b_k)\cup\{\nu_k\}\cup(c_k,d_k]$ will be given by the transformation $f:[0,1]\to[a_k,b_k)\cup\{\nu_k\}\cup(c_k,d_k]$ described by

$$f(x) = \begin{cases} (b_k-a_k)\frac{x}{e}+a_k & \text{if } x\in[0,e),\\ \nu_k & \text{if } x=e,\\ d_k-\frac{(1-x)(d_k-c_k)}{1-e} & \text{otherwise} \end{cases}$$

for $0\le a_k\le b_k\le e\le c_k\le d_k\le 1$, $\nu_k\in[b_k,c_k]$. Then

$$U_{\nu_k}^{a_k,b_k,c_k,d_k}(x,y) = f(U(f^{-1}(x),f^{-1}(y))). \tag{1.19}$$

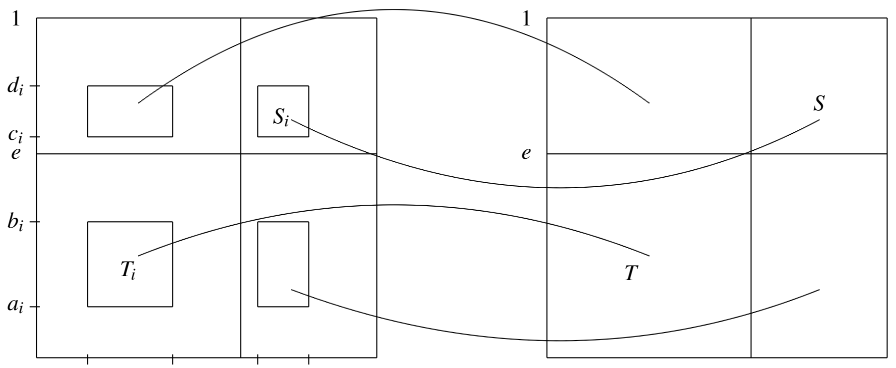

Fig. 1.8 Isomorphism given by (1.19)

In the case that $a_k = b_k$ ($c_k = d_k$) the corresponding summand is isomorphic to a t-conorm (t-norm). Note, that if $U_k(x,y) = e_k$ for some $x\neq e$, $y\neq e_k$ then $[0,e_k)\cup(e_k,1]$ is not closed under U_k and thus in order to preserve associativity ν_k has to be an annihilator of U restricted to $[b_k,c_k]^2$.

In order to obtain the monotonicity of the resulting operation the order of summand have to be compatible with the standard order on $[0,e]$ and reversed with respect to the standard order on $[e,1]$. This approach distinguishes the construction of ordinal sum of uninorms presented in [200] from the one presented earlier in [268, 102, 73] and others, where an order consistent with the natural order and two types of sums were used: ordinal sum and dual ordinal sum (see Theorem 1.22).

So, $k_1 \prec k_2$ for $k_1, k_2 \in K$ implies $b_{k_1} \le a_{k_2}, d_{k_2} \le c_{k_1}$, i.e., $[a_{k_2}, d_{k_2}]^2 \subset [b_{k_1}, c_{k_1}]^2 \subset [a_{k_1}, d_{k_1}]^2$.

Let us denote $K_* = \{k \in K : (a_k, b_k) \neq \emptyset\}$ and $K^* = \{k \in K : (c_k, d_k) \neq \emptyset\}$ then $B_1 = \bigcup_{k \in K}[a_k, b_k] \setminus \bigcup_{k \in K}[a_k, b_k) = \{b_k : k \in K\} \setminus \{a_k : k \in K_*\}$, $C_1 = \bigcup_{k \in K}[c_k, d_k] \setminus \bigcup_{k \in K}(c_k, d_k] = \{c_k : k \in K\} \setminus \{d_k : k \in K^*\}$. Since K is assumed to be countable then every $b \in B_1 \setminus \{e\}$ is an accumulation point of $\{a_k : k \in K_*\}$ ($c \in C_1 \setminus \{e\}$). Let us denote $B_2 = B_1 \setminus \{e\}$, $C_2 = C_1 \setminus \{e\}$ and define functions $g : B_2 \to [e, 1]$, $h : C_2 \to [0, e]$, such that if for $b \in B_2$ we have $b = \lim_{i \to \infty} a_{k_i}$ for $k_i \in K_*$, then

$$g(b) = \lim_{i \to \infty} d_{k_i}. \tag{1.20}$$

Similarly, for $c \in C_2$ we have $c = \lim_{i \to \infty} d_{k_i}$ for $k_i \in K^*$, then

$$h(c) = \lim_{i \to \infty} a_{k_i}. \tag{1.21}$$

If $g(b) \notin C_2$ for some $b \in B_2$ ($h(c) \notin B_2$ for some $c \in C_2$) then the value of $U(b, g(b))$ ($U(c, h(c))$) follows from the monotonicity of U. Therefore we have only to consider the case when $g(b) \in C_2$ ($h(c) \in B_2$).

Theorem 1.23 ([199]). *Let $e \in [0,1]$ and let K be an index set which is finite or countably infinite, $\{(a_k, b_k)\}_{k \in K}$ be a disjoint family of open subintervals (which can be also empty) of $[0, e]$, such that $\bigcup_{k \in K}[a_k, b_k] = [0, e]$. Similarly, let $\{(c_k, d_k)\}_{k \in K}$ be a disjoint family of open subintervals (which can be also empty) of $[e, 1]$, such that $\bigcup_{k \in K}[c_k, d_k] = [e, 1]$. Let further these two families be anti-comonotone, i.e., $b_k \le a_i$ if and only if $c_k \ge d_i$ for all $i, k \in K$. We will denote $K_* = \{k \in K : (a_k, b_k) \neq \emptyset\}$ and $K^* = \{k \in K : (c_k, d_k) \neq \emptyset\}$. Let us assume that there are given a family of proper uninorms $\{U_k\}_{k \in K_* \cap K^*}$ on $[0,1]^2$, a family of t-norms $\{U_k\}_{k \in K_* \setminus K^*}$ on $[0,1]^2$ and a family of t-conorms $\{U_k\}_{k \in K^* \setminus K_*}$ on $[0,1]^2$. Denote $B_1 = \{b_k : k \in K\} \setminus \{a_k : k \in K_*\}$, and $C_1 = \{c_k : k \in K\} \setminus \{d_k : k \in K^*\}$ and let $B = \{b \in B_1 \setminus \{e\} : g(b) \in C_1\}$. $C = \{c \in C_1 \setminus \{e\} : h(c) \in B_1\}$, where the functions g and h are given by* (1.20) *and* (1.21)*. Further assume a function $n : B \to B \cup C$ given for all $b \in B$ by*

$$n(b) \in \{b, g(b)\}.$$

Let the ordinal sum $U^e = (\langle a_k, b_k, c_k, d_k, U_k \rangle : k \in K)^e$ be given by

$$U^e(x,y) = \begin{cases} y & if\ x = e, \\ x & if\ y = e, \\ (U_k)_{v_k}^{a_k,b_k,c_k,d_k}(x,y) & if\ (x,y) \in ([a_k,b_k) \cup (c_k,d_k])^2,\ k \in K_* \cap K^*, \\ (U_k)^{a_k,b_k}(x,y) & if\ (x,y) \in ([a_k,b_k) \cup (c_k,d_k])^2,\ k \in K_* \setminus K^*, \\ (U_k)^{c_k,d_k}(x,y) & if\ (x,y) \in ([a_k,b_k) \cup (c_k,d_k])^2,\ k \in K^* \setminus K_*, \\ x & if\ y \in [b_k,c_k], x \in [a_k,d_k] \setminus [b_k,c_k], k \in K_* \cup K^*, \\ y & if\ x \in [b_k,c_k], y \in [a_k,d_k] \setminus [b_k,c_k], k \in K_* \cup K^*, \\ \min(x,y) & if\ (x,y) \in [b,c]^2 \setminus ((b,c)^2 \cup \{(b,c),(c,b)\}), \\ & where\ b \in B, c = g(b), x+y < c+b, \\ \max(x,y) & if\ (x,y) \in [b,c]^2 \setminus ((b,c)^2 \cup \{(b,c),(c,b)\}), \\ & where\ b \in B, c = g(b), x+y > c+b, \\ n(b) & if\ (x,y) = (b,c)\ or\ (x,y) = (c,b), b \in B, c = g(b), \\ \min(x,y) & if (x,y) \in \{b\} \times [b,c] \cup [b,c] \times \{b\}\ and \\ & b \in B_1 \setminus (B \cup \{e\}),\ c = g(b), \\ \max(x,y) & if (x,y) \in \{c\} \times [b,c] \cup [b,c] \times \{c\}\ and \\ & c \in C_1 \setminus (C \cup \{e\}),\ b = h(c), \end{cases}$$

where $\nu_k = c_k$ ($\nu_k = b_k$) if there exists an $i \in K$ such that $b_k = a_i$, $c_k = d_i$ and U_i is disjunctive (conjunctive) and $\nu_k = n(b_k)$ if $b_k \in B$, $\nu_k = b_k$ if $b_k \in B_1 \setminus B$, $\nu_k = c_k$ if $c_k \in C_1 \setminus C$, and $(U_k)_{\nu_k}^{a_k,b_k,c_k,d_k}$ is given by the formula (1.19)*, $(U_k)^{a_k,b_k}$ ($(U_k)^{c_k,d_k}$) is a linear transformation of U_k to $[a_k,b_k]^2$ ($[c_k,d_k]^2$). Then U^e is a uninorm.*

1.9 Representable uninorms

In Sections 1.4 and 1.5 Archimedean t-norms and t-conorms for which there is an additive generator have been reminded. Here we present an analogous construction for uninorm.

Theorem 1.24 ([102, 98, 153]). *Let $U : [0,1]^2 \to [0,1]$ be a binary operation and $e \in (0,1)$. The following statements are equivalent:*

(i) U is a uninorm with the neutral element e that is strictly increasing on $(0,1)^2$ and continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$.

(ii) There exists a strictly increasing bijection $u : [0,1] \to [-\infty,+\infty]$ with $u(e) = 0$ such that for all $(x,y) \in [0,1]^2$ it holds that

$$U(x,y) = u^{-1}(u(x)+u(y)), \tag{1.22}$$

where in case of a conjunctive uninorm U, we adopt the convention $(+\infty)+(-\infty) = -\infty$, while in case of a disjunctive uninorm, we adopt the convention $(+\infty)+(-\infty) = +\infty$.

(iii) Operation U is a uninorm with the neutral element e which is continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$.

(iv) U is a uninorm with the neutral element e where both T_U and S_U are strict and there exist $(x_0, y_0) \in intA(e)$ *such that* $\min(x_0, y_0) < U(x_0, y_0) < \max(x_0, y_0)$.

If representation (1.22) *holds, then the function u is uniquely determined by U up to a positive multiplicative constant, and it is called an additive generator of the uninorm U.*

The family of all uninorm described in Theorem 1.24 will be denoted by $\mathscr{U}_{rep}$ and uninorms belonging to this family are called representable uninorms.

The underlying t-norm T_U and underlying t-conorm S_U for a representable uninorm U with additive generator u, are strict and have additive generator t given by $t(x) = -u(ex)$, and s given by $s(x) = u(e + (1-e)x)$.

Suppose now that the underlying t-norm T and t-conorm S of a uninorm with the neutral element $e \in (0,1)$, are strict t-norm and t-conorm with additive generators t and s, respectively. We can construct a representable uninorm U with the neutral element e such that its underlying t-norm and t-conorm are T and S. Indeed, define a function $u : [0,1] \to [-\infty, +\infty]$ by

$$u(x) = \begin{cases} -t\left(\frac{x}{e}\right) & if\ x \in [0,e], \\ s\left(\frac{x-e}{1-e}\right) & if\ x \in (e,1]. \end{cases} \tag{1.23}$$

The inverse of this function is given by

$$u^{-1}(x) = \begin{cases} et^{-1}(-x) & if\ x \leq e, \\ e + (1-e)s^{-1}(x) & if\ x > e. \end{cases}$$

Example 1.13 (cf. [143] Example 10.12). Let $\lambda \in (0, \infty)$, $u^\lambda : [0,1] \to [-\infty, \infty]$ be defined by the formula

$$u^\lambda(x) = ln \frac{\lambda x}{1-x}$$

Then construction (1.22) leads to the uninorm

$$U^\lambda(x,y) = \begin{cases} 0 \text{ or } 1 & \text{if } (x,y) \in \{(0,1),(1,0)\} \\ \frac{\lambda xy}{\lambda xy + (1-x)(1-y)} & \text{otherwise} \end{cases}$$

In this case the neutral element of U is $e = \frac{1}{1+\lambda}$, the underlying t-norm is the Hamacher t-norm $T^H_{\frac{\lambda+1}{\lambda}}$ and underlying t-conorm is the Hamacher t-conorm $S^H_{\lambda+1}$.

Example 1.14. In the construction (1.23) we can use the additive generator of the Hamacher family of t-norm and t-conorm with parameter λ (for more detail see [143] pp. 105–107, 322–323) and obtain the additive generator in the form

$$u_\lambda(x) = \begin{cases} \frac{x-e}{x} & \text{if } \lambda = 0 \text{ and } x \leq e, \\ \frac{x-e}{1-x} & \text{if } \lambda = 0 \text{ and } e < x, \\ -\ln \frac{\lambda e + x - \lambda x}{x} & \text{if } \lambda > 0 \text{ and } x \leq e, \\ \ln \frac{1 - \lambda e - x + \lambda x}{1-x} & \text{if } \lambda > 0 \text{ and } e < x \end{cases}$$

with the inverse function

$$u_\lambda^{-1}(x) = \begin{cases} \frac{e}{1-x} & \text{if } \lambda = 0 \text{ and } x < 0, \\ \frac{x+e}{1+x} & \text{if } \lambda = 0 \text{ and } x \geq 0, \\ \frac{\lambda e}{E^{-x}+\lambda-1} & \text{if } \lambda > 0 \text{ and } x < 0, \\ \frac{e+(1-e)(1-E^x)}{1-\lambda-E^x} & \text{if } \lambda > 0 \text{ and } x \geq 0. \end{cases}$$

If we fix for example $\lambda = 2$ (the case when the underlying t-norm and t-conorm are Einstein t-norm and t-conorm, respectively) and $e = 0.5$ then using formula (1.22) we obtain the uninorm (see Figure 1.9, left part)

$$U(x,y) = \begin{cases} 1 & if\ (x,y) \in \{(0,1),(1,0)\} \\ \frac{xy}{1-x-y+2xy} & otherwise \end{cases}$$

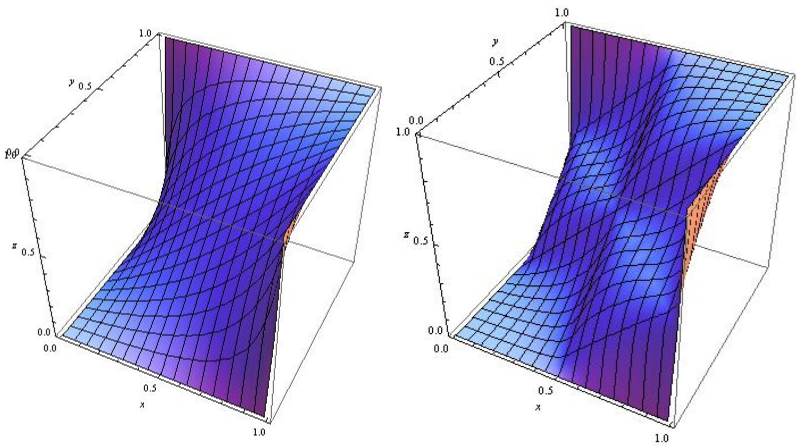

Fig. 1.9 Representable uninorms with $e = 0.5$ and the Hamacher underlying operators

Remark 1.6. Note that in Example 1.13 we obtain independent underlying t-norm and t-conorm from family Hamacher operations. Only in the case $\lambda = 1$ we obtain dual operations (product t-norm and t-conorm). Moreover, the parameter λ strictly depends on the neutral element of uninorm. Whereas in Example 1.14 underlying t-norm and t-conorm are dual. Thus, we can scale both the neutral element and the quantity of the underlying operations (see Figure 1.9, right part).

Remark 1.7. In general, using t-norm and t-conorm generators to construct pseudo-continuous uninorms does not always produce the desired results. For instance, taking the Łukasiewicz t-norm T_L and t-conorm S_L together with their additive generators specified by $t(x) = 1 - x$ and $s(x) = x$, and fixing the neutral element $e = 0.5$ we have (see Figure 1.10)

$$U(x,y) = \min(1, \max(0, x+y-0.5)).$$

This operation is not associative, but increasing and with the neutral element $e = 0.5$.

In general, if we relax the condition $u(0) = -\infty$, $u(1) = +\infty$, the associativity condition will no longer be satisfied (if $u(0) < 0$ and $u(1) > 0$) and if we relax the strict monotonicity of u then the neutral element will be lost (see [200]).

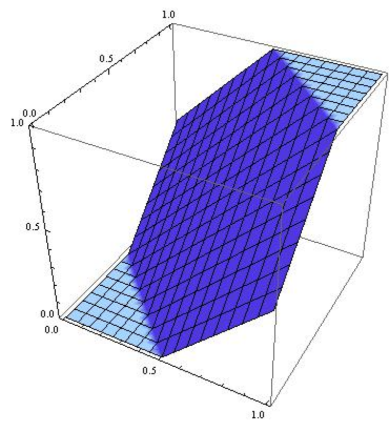

Fig. 1.10 Operation with $e = 0.5$ using the Łukasiewicz underlying t-norm and t-conorm

1.10 Continuous uninorms in the open unit square

In the next theorem visualized in Figure 1.11 there is given the characterization of uninorms which are continuous on the open unit square. We will denote the family of all such uninorms by $\mathscr{U}_{cos}$. In fact, the uninorms are continuous beyond the boundary points in the interval $[c,a]$ or $[b,p]$.

Lemma 1.5 ([78]). *If U is a uninorm with neutral element $e \in (0,1)$ continuous in $(0,1)^2$ then T_U and S_U are continuous.*

Theorem 1.25 ([124, 78]). *If U is a uninorm with neutral element $e \in (0,1)$ continuous in $(0,1)^2$ then one of the following two cases holds:*

1. *There exists $a \in [0,e[$, $c \in [0,a]$, two continuous t-norms T_1 and T_2 and a representable uninorm R such that U can be represented as*

$$U(x,y)=\begin{cases} cT_1(\frac{x}{c},\frac{y}{c}) & if\ x,y\in[0,c],\\ c+(a-c)T_2(\frac{x-c}{a-c},\frac{y-c}{a-c}) & if\ x,y\in[c,a],\\ a+(1-a)R(\frac{x-a}{1-a},\frac{y-a}{1-a}) & if\ x,y\in[a,1),\\ 1 & if\ \min(x,y)\in(c,1]\ and\ \max(x,y)=1,\\ 1\ or\ c & if\ (x,y)\in\{(c,1),(1,c)\},\\ \min(x,y) & otherwise.\end{cases} \tag{1.24}$$

2. *There exists $b\in(e,1]$, $p\in[b,1]$, two continuous t-conorms S_1 and S_2 and a representable uninorm R such that U can be represented as*

$$U(x,y)=\begin{cases} bR(\frac{x}{b},\frac{y}{b}) & if\ x,y\in(0,b],\\ b+(p-b)S_1(\frac{x-b}{p-b},\frac{y-b}{p-b}) & if\ x,y\in[b,p],\\ p+(1-p)S_2(\frac{x-p}{1-p},\frac{y-p}{1-p}) & if\ x,y\in[p,1],\\ 0 & if\ \max(x,y)\in[0,p)\ and\ \min(x,y)=0,\\ 0\ or\ p & if\ (x,y)\in\{(p,0),(0,p)\},\\ \max(x,y) & otherwise.\end{cases} \tag{1.25}$$

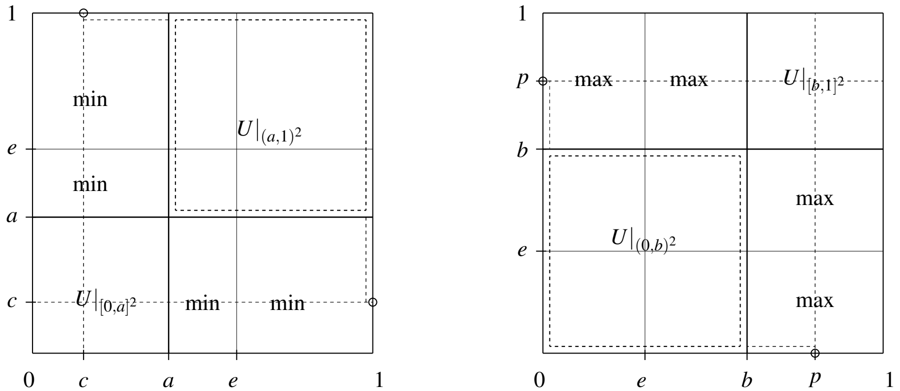

Fig. 1.11 The uninorm which is continuous in the open unit square with $a>0$ (left) and $b<1$ (right)

1.11 Idempotent uninorms

In the paper [52] Czogała and Drewniak described some class of operations which was later named idempotent uninorms. Next in [57, 72, 173, 174, 235] idempotent uninorms were characterized, finally using the terminology of Id-symmetrical functions. We will denote the family of all idempotent uninorms by $\mathscr{U}_{id}$. Let us recall some definitions about this topic, that can be found in [62].

Definition 1.14 ([62]). Let $g : [0,1] \to [0,1]$ be any decreasing function and let G be the graph of g, that is

$$G = \{(x, g(x)) : x \in [0,1]\}.$$

For any discontinuity point s of g, let us denote by s^- and s^+ the corresponding one sided limits, that are $s^- = \lim_{x\to s^-} g(x)$ and $s^+ = \lim_{x\to s^+} g(x)$.

Then, we define the *completed graph* of g, as the set

$$F_g = G \cup \{(0,y) : y > g(0)\} \cup \{(1,y) : y < g(1)\} \cup \{(s,y) : s^- \le y \le s^+\}.$$

Definition 1.15 ([62]). A subset F of $[0,1]^2$ is said to be *Id-symmetrical* if for all $(x,y) \in [0,1]^2$ it holds that

$$(x,y) \in F \iff (y,x) \in F\,.$$

The above definition expresses that a subset F of $[0,1]^2$ is symmetrical w.r.t. the diagonal of the unit square. A similar notion of symmetry is introduced for decreasing functions (see [62]) as follows.

Definition 1.16 ([62]). A decreasing function $g : [0,1] \to [0,1]$ is called *Id-symmetrical* if its completed graph F_g is Id-symmetrical.

Definition 1.17. A uninorm U with the neutral element $e \in (0,1)$ is called locally internal in a region $R \subseteq [0,1]^2$ if it satisfies $U(x,y) \in \{x,y\}$ for all $(x,y) \in R$.
If $R = [0,1]^2$ then U is called locally internal. Let us denote by $\mathscr{U}_{loc}$ the family of all locally internal uninorms.

Theorem 1.26 ([174]). *Operation U is a locally internal uninorm if and only if it is an idempotent uninorm.*

Theorem 1.27 ([235]). *Let $e \in (0,1)$. U is an idempotent uninorm with the neutral element e if and only if there exists a decreasing, Id-symmetrical function $g : [0,1] \to [0,1]$ with the fixed point e such that U is given by*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x) \text{ and } x < g(g(x)),\\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x) \text{ and } x > g(g(x)),\\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g(g(x)), \end{cases} \tag{1.26}$$

being commutative on the set of points $(x, g(x))$ such that $x = g^2(x)$.

We can also characterize locally internal uninorms using ordinal sum construction.
For semigroups that are defined on singletons we will further use an operation

$$Id : \{x\}^2 \to \{x\} \text{ given by } Id(x,x) = x.$$

Theorem 1.28 ([198]). *Let $U : [0,1]^2 \to [0,1]$ be an internal uninorm. Then $([0,1],U)$ is an ordinal sum of singleton semigroups $(\{x\}, Id)$ for all $x \in [0,1]$.*

Theorem 1.29 ([200, 198]). *Let P be an index set isomorphic with $[0,1]$ via the isomorphism $i : P \to [0,1]$. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Let $e \in [0,1]$ and let $\preceq$ be a linear order on P. Then the ordinal sum of $\{(X_p, Id)\}_{p\in P}$ with the linear order $\preceq$ is an internal uninorm with the neutral element e if and only if the following two conditions are fulfilled:*

(i) $p_1 \prec p_2$ for all $p_1,\ p_2 \in P$ such that $X_{p_1} = \{x_1\}$, $X_{p_2} = \{x_2\}$, $x_1 < x_2$ and $x_1,\ x_2 \in [0,e]$,
(ii) $p_1 \prec p_2$ for all $p_1,\ p_2 \in P$ such that $X_{p_1} = \{y_1\}$, $X_{p_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e,1]$.

1.12 Locally internal uninorms on $A(e)$

A generalization of idempotent uninorms are locally internal uninorms on the subset $A(e)$ whose family we will denote by $\mathscr{U}_{locA}$. Description of such uninorms can be found in [52, 72, 79, 235]. Here, we will first give a general form of this type of uninorm, and then we will characterize some classes of these uninorms.

Theorem 1.30. *If U is a uninorm with the neutral element $e \in (0,1)$ and locally internal in $A(e)$ then there exists a decreasing function $g : [0,1] \to [0,1]$ with $g(e) = e$ which is Id-symmetrical, in such a way that U is given in the region $A(e)$ by*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x) \text{ and } x < g(g(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x) \text{ and } x > g(g(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g(g(x)). \end{cases} \tag{1.27}$$

From now on, the function g obtained in the previous theorem will be called the *associated function* of the uninorm U. In order to characterize all locally internal uninorms with the continuous underlying operations, we will first investigate how can be described such a decreasing and Id-symmetrical function g in these cases.

1.12.1 Properties of g

First of all, note that the condition of symmetry is quite intuitive but difficult to manage from a mathematical point of view. In this way, it was proved in [235] (see also [76, 79]) the following equivalence.

Proposition 1.1 ([235]). *Let $g : [0,1] \to [0,1]$ be a decreasing function with the fixed point $e \in (0,1)$. Then g is Id-symmetrical if and only if g satisfies the following two conditions:*

i) $\inf\{y \mid g(y) = g(x)\} \leq g(g(x)) \leq \sup\{y \mid g(y) = g(x)\}$

ii) g is constant, say $g(x) = s$ in the interval (p,q) with $p < q$, where

$$p = \inf\{x \in [0,1] \mid g(x) = s\} \qquad \text{and} \qquad q = \sup\{x \in [0,1] \mid g(x) = s\},$$

if and only if, $s \in (0,1)$ is a discontinuity point of g or $s \in \{0,1\}$ and it is satisfied that

$$p = \begin{cases} s^+ & \text{if } s < 1, \\ 0 & \text{if } s = 1 \end{cases} \quad \text{and} \quad q = \begin{cases} s^- & \text{if } s > 0, \\ 1 & \text{if } s = 0. \end{cases}$$

Remark 1.8. Note that from condition (i) in the previous proposition, it is directly derived that if g is strictly decreasing and continuous on an interval $(a,b) \subseteq [0,1]$ then $g((a,b)) = (c,d) \subseteq [0,1]$, g is also strictly decreasing and continuous on (c,d) and $g^2(x) = g(g(x)) = x$ for all $x \in (a,b) \cup (c,d)$.

On the other hand, note that cases $s \in \{0,1\}$ in condition (ii) can be points of discontinuity or not. In Figure 1.12, two examples of g functions are depicted.

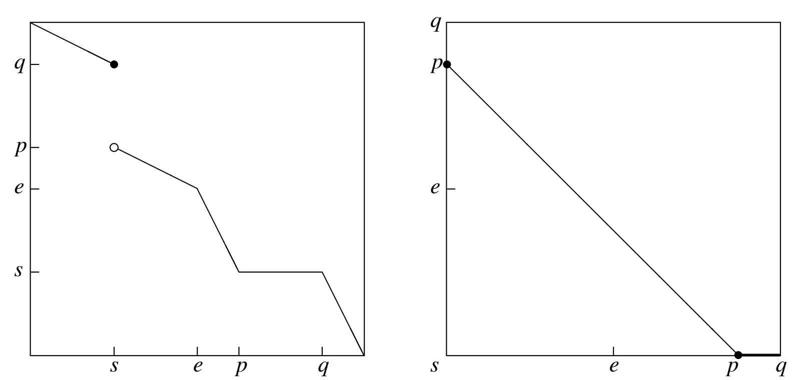

Fig. 1.12 Examples of functions g with $s \in (0,1)$ (left) and $s = 0$ (right)

Now, let us note that all elements in the image of function g must be idempotent elements of the uninorm U, by distinguishing some possible cases.

Proposition 1.2. *Let U be a uninorm with the neutral element $e \in (0,1)$, locally internal in $A(e)$ and let g be its associated function. If g is strictly decreasing and continuous on (a,b), then:*

- *$g(x)$ is an idempotent element of U for all $x \in (a,b)$, and*
- *x is an idempotent element of U for all $x \in [a,b]$.*

Proposition 1.3. *Let U be a uninorm with the neutral element $e \in (0,1)$, locally internal in $A(e)$ and let g be its associated function. If g is constant, say $g(x) = s$ in the interval (a,b) with $a = \inf\{x \in [0,1] : g(x) = s\} < b = \sup\{x \in [0,1] : g(x) = s\}$, then s, a and b are idempotent elements of U.*

Remark 1.9. Note that in Proposition 1.3, when g is constant in an interval $[a,b]$, the extremal points must be idempotent, but the elements $x \in (a,b)$ may be idempotent or not.

Proposition 1.4. *Let U be a uninorm with the neutral element $e \in (0,1)$, locally internal in $A(e)$ and let g be its associated function. If x is a discontinuity point of g then both x and $g(x)$ are idempotent elements of U.*

Joining all the propositions above we trivially obtain the following result, which will be crucial in the later characterization theorems.

Theorem 1.31. *Let U be a uninorm locally internal in $A(e)$ such that T_U and S_U are continuous. Then for all $x \in [0,1]$, $g(x)$ is an idempotent element of U.*

To finish this subsection, we prove some additional properties of function g, depending on the structure of the continuous underlying operators of U. From the classification theorem of continuous t-norms and t-conorms (see Theorem 1.19 and Theorem 1.21), we know that if T_U and S_U are continuous they are given by ordinal sums.

Before giving the characterization theorem in this general case let us note first the following remark.

Remark 1.10. Let U be a uninorm with the neutral element $e \in (0,1)$ and the underlying t-norm T and t-conorm S. Then the restriction of U to the square $[0,e]^2$ is given by an ordinal sum of t-norms of the form $(\langle a_i, b_i, T_i\rangle)_{i\in I}$ if and only if T is an ordinal sum of t-norms of the form $T = \left(\left\langle \frac{a_i}{e}, \frac{b_i}{e}, T_i\right\rangle\right)_{i\in I}$. Similarly, the restriction of U to the square $[e,1]^2$ is given by an ordinal sum of t-conorms of the form $\left(\langle c_j, d_j, S_j\rangle\right)_{j\in J}$ if and only if S is an ordinal sum of t-conorms of the form $S = \left(\left\langle \frac{c_j-e}{1-e}, \frac{d_j-e}{1-e}, S_j\right\rangle\right)_{j\in J}$.

In order to avoid the scaling mentioned in the previous remark, instead of dealing with uninorm U with T and S given by concrete ordinal sums, we will equivalently talk about uninorms U such that their restrictions to the square $[0,e]^2$ and $[e,1]^2$ are given by ordinal sums of the form $(\langle a_i, b_i, T_i\rangle)_{i\in I}$ and $(\langle c_j, d_j, S_j\rangle)_{i\in J}$, respectively.

Lemma 1.6. *Let U be a uninorm with the neutral element $e \in (0,1)$, locally internal in $A(e)$ and let g be its associated function. Suppose that the restriction of U to the square $[0,e]^2$ is given by an ordinal sum of Archimedean t-norms of the form $(\langle a_i, b_i, T_i\rangle)_{i\in I}$. Then*

i) Function g is constant on each interval (a_i,b_i), i.e. $g((a_i,b_i)) = s_i$ for all $i \in I$.
ii) If T_i is nilpotent for some $i \in I$, then $g(a_i) = s_i$ and

$$U(x,s_i) = s_i \ \text{ for all } \ x \in [a_i,b_i) \qquad \text{or} \qquad U(x,s_i) = x \ \text{ for all } \ x \in [a_i,b_i).$$

Lemma 1.7. *Let U be a uninorm with the neutral element $e \in (0,1)$, locally internal in $A(e)$ and let g be its associated function. Suppose that the restriction of U to the square $[e,1]^2$ is given by an ordinal sum of Archimedean t-conorms of the form $\langle (c_i, d_i, S_i)\rangle_{i\in I}$. Then*

i) Function g is constant on each interval (c_i, d_i), i.e. $g((c_i, d_i)) = r_i$ for all $i \in I$.
ii) If S_i is nilpotent for some $i \in I$, then $g(c_i) = r_i$ and

$$U(x, r_i) = r_i \text{ for all } x \in (c_i, d_i] \qquad \text{or} \qquad U(x, r_i) = x \text{ for all } x \in (c_i, d_i].$$

1.12.2 The characterization theorem

In this subsection, we will give a theorem characterizing uninorms locally internal on $A(e)$ in general form.

Theorem 1.32. *Let $U : [0,1] \to [0,1]$ be a binary operation and $e \in (0,1)$. U is a uninorm with the neutral element e, locally internal in $A(e)$ if and only if there exists a decreasing and Id-symmetrical function $g : [0,1] \to [0,1]$ with $g(e) = e$ satisfying the following conditions:*

i) $g(x) \in [e,1] \setminus \cup_{j\in J} (c_j, d_j)$ for all $x \in [0,e]$ and, similarly, $g(x) \in [0,e] \setminus \cup_{i\in I} (a_i, b_i)$ for all $x \in [e,1]$.
ii) For all $i \in I$, g is constant in the interval (a_i, b_i), say $g(x) = s_i$ for all $x \in (a_i, b_i)$, and also g is constant in the interval (c_j, d_j) for all $j \in J$, say $g(x) = r_j$ for all $x \in (c_j, d_j)$.
iii) $g(b_i) = s_i$ and ($U(x, s_i) = x$ on $(a_i, b_i]$ or $U(x, s_i) = s_i$ on $(a_i, b_i]$), or $U(b_i, s_i) = s_i$ and T_i has the neutral element b_i.
iv) $g(c_j) = r_j$ and ($U(x, r_j) = r_j$ on $[c_j, d_j)$ or $U(x, r_j) = x$ on $[c_j, d_j)$) or $U(c_j, r_j) = r_j$ and S_j has the neutral element c_j.
v) If T_i has the zero element divisors (where the zero element is equal to a_i), then $g(a_i) = s_i$ and ($U(x, s_i) = s_i$ on $[a_i, b_i)$ or $U(x, s_i) = x$ on $[a_i, b_i)$).
vi) If S_j has zero divisors (where the zero element is equal d_j), then $g(d_j) = r_j$ and ($U(x, r_j) = x$ on $(c_j, d_j]$ or $U(x, r_j) = r_j$ on $(c_j, d_j]$).
vii) If b_i is the zero element divisor of operation T_i, then $g(a_i) = g(b_i) = s_i$ and ($U(x, s_i) = s_i$ on $[a_i, b_i]$ or $U(x, s_i) = x$ on $[a_i, b_i]$).
viii) If c_j is the zero divisor of operation S_j, then $g(c_j) = g(d_j) = r_j$ and ($U(x, r_j) = x$ on $[c_j, d_j]$ or $U(x, r_j) = r_j$ on $[c_j, d_j]$)

such that U is given by the ordinal sum $(\langle a_i, b_i, T_i\rangle)_{i\in I}$ in $[0,e]^2$, by $(\langle c_j, d_j, S_j\rangle)_{j\in J}$ in $[e,1]^2$, and by the expression:

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x) \text{ and } x < g(g(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x) \text{ and } x > g(g(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g(g(x)), \end{cases} \tag{1.28}$$

in $A(e)$, being commutative on the set of points $(x, g(x))$ such that $x = g^2(x)$.

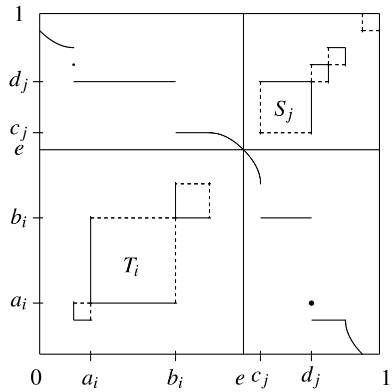

Fig. 1.13 Structure of uninorms locally internal on $A(e)$

Proof. The proof is analogous to the proof of Theorem 6 from paper [78] with the difference that instead of using the Theorem characterizing idempotent uninorms from paper [174] (which turned out to contain some errors) we should use Theorem 1.27 from the paper [235]. □

1.13 Characterization theorems for uninorms with the continuous underlying operators

We want to deal now with the characterization family of all uninorms with the given continuous underlying operators (denoted by $\mathcal{U}_{cou}$). We will do this by dividing our reasoning in several particular cases depending on how the corresponding t-norm T and t-conorm S are. Specifically, we study the cases where the component operations are Archimedean, idempotent, and finally given as an ordinal sum of previously considered operations.

1.13.1 Case T_U and S_U idempotent

This case is already known since it corresponds to the case of idempotent uninorms. It is well known that all idempotent uninorms must be in fact locally internal. Thus, all possible idempotent uninorms were already characterized in Section 1.11 by Theorem 1.27 (we include the result here for the sake of completeness).

1.13.2 Case T_U Archimedean, S_U idempotent

This case has been partially studied in [82], for the more general case when the operation U is not commutative, dealing with properties on U. Now we present a general characterization. Note that the particular case when T_U is nilpotent has been recently characterized [222] through a completely different method, using geometrical reasonings. Some results can be found also in [155], [156] and [154]

Theorem 1.33. *A binary operation $U : [0,1]^2 \to [0,1]$ is a uninorm U such that T_U is a continuous Archimedean t-norm and $S_U = \max$ if, and only if, there exist $s,t \in [e,1]$ such that $s \leq t$ and U is given by one of the following cases*

$$\text{(i) } U(x,y) = \begin{cases} eT_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ \min(x,y) & \text{if } 0 < \min(x,y) < e < \max(x,y) \leq s, \\ & \text{or } 0 = \min(x,y) < e < \max(x,y) \leq t, \\ \max(x,y) & \text{otherwise}, \end{cases}$$

$$\text{(ii) } U(x,y) = \begin{cases} eT_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ \min(x,y) & \text{if } 0 < \min(x,y) < e < \max(x,y) \leq s, \\ & \text{or } 0 = \min(x,y) < e < \max(x,y) < t, \\ \max(x,y) & \text{otherwise}, \end{cases}$$

$$\text{(iii) } U(x,y) = \begin{cases} eT_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ \min(x,y) & \text{if } 0 < \min(x,y) < e < \max(x,y) < s, \\ & \text{or } 0 = \min(x,y) < e < \max(x,y) < t, \\ \max(x,y) & \text{otherwise}, \end{cases}$$

$$\text{(iv) } U(x,y) = \begin{cases} eT_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ \min(x,y) & \text{if } 0 < \min(x,y) < e < \max(x,y) < s, \\ & \text{or } 0 = \min(x,y) < e < \max(x,y) \leq t, \\ \max(x,y) & \text{otherwise}, \end{cases}$$

with the restriction that when T_U is nilpotent it must be $s = t$ and expressions (ii) and (iv) can not be fulfilled.

Proof. It was proved in [82] that all uninorms with the underlying Archimedean continuous t-norm T_U and $S_U = \max$ must be in fact locally internal in $A(e)$, see also [222]. Thus, then there exists a decreasing Id-symmetrical function $g : [0,1] \to [0,1]$ with the fixed point e such that U is given by (1.27) in $A(e)$.

Since the set of idempotent elements of U is a set $\{0\} \cup [e,1]$, then using Theorem 1.31, we obtain, that $g(x) \in \{0\} \cup [e,1]$ for all $x \in [0,1]$. We get directly from here $g(x) \in \{0,e\}$ for $x \in [e,1]$, and therefore by Proposition 1.1 g is constant in $(0,e)$. Taking $s = g((0,e))$, $t = g(0)$ and using the assumption that g is Id-symmetrical, we have two cases for g (graphs are drawn in Figure 1.14):

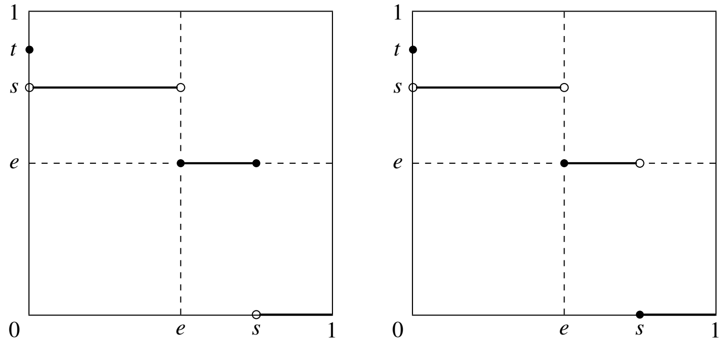

Fig. 1.14 Functions g_1 (left) and g_2 (right) for T_U Archimedean and $S_U = \max$

$$g_1(x) = \begin{cases} t & \text{if } x = 0, \\ s & \text{if } x \in (0,e), \\ e & \text{if } e \le x \le s, \\ 0 & \text{if } x > s, \end{cases} \qquad g_2(x) = \begin{cases} t & \text{if } x = 0, \\ s & \text{if } x \in (0,e), \\ e & \text{if } e \le x < s, \\ 0 & \text{if } x \ge s. \end{cases}$$

The function g_1 leads to the cases (i) and (ii), while the function g_2 leads to the cases (iii) and (iv). Moreover, when T_U is nilpotent applying Lemma 1.6-ii), it must be $g(0) = t = s$ and cases (ii) and (iv) can not be fulfilled.

Conversely, it is a straightforward computation to see that it is a uninorm locally internal in $A(e)$. □

1.13.3 Case T_U continuous, S_U idempotent

Theorem 1.34 ([156]). *Let U be a uninorm with the neutral element $e \in (0,1)$. If T_U is a continuous t-norm, i.e., it is an ordinal sum of Archimedean t-norm and $S_U = S_M$, then $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$.*

Because of this properties the full characterization will be completed in Section 1.13.7.

1.13.4 Case T_U idempotent, S_U Archimedean

This case is very similar to the previous one (see Subsection 1.13.2), and it can be derived by duality.

Theorem 1.35 (cf. [90]). *A binary operation $U : [0,1]^2 \to [0,1]$ is a uninorm U such that $T_U = \min$ and S_U is a continuous Archimedean t-conorm if, and only if, there exist $s,t \in [0,e]$ such that $t \le s$ and U is given by one of the following cases:*

$$(i)\ U(x,y)=\begin{cases} e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y)\in[e,1]^2,\\ \max(x,y) & \text{if } 1>\max(x,y)>e>\min(x,y)\geq s\\ & \text{or } 1=\max(x,y)>e>\min(x,y)\geq t,\\ \min(x,y) & \text{otherwise},\end{cases}$$

$$(ii)\ U(x,y)=\begin{cases} e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y)\in[e,1]^2,\\ \max(x,y) & \text{if } 1>\max(x,y)>e>\min(x,y)\geq s\\ & \text{or } 1=\max(x,y)>e>\min(x,y)>t,\\ \min(x,y) & \text{otherwise},\end{cases}$$

$$(iii)\ U(x,y)=\begin{cases} e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y)\in[e,1]^2,\\ \max(x,y) & \text{if } 1>\max(x,y)>e>\min(x,y)>s\\ & \text{or } 1=\max(x,y)>e>\min(x,y)>t,\\ \min(x,y) & \text{otherwise},\end{cases}$$

$$(iv)\ U(x,y)=\begin{cases} e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y)\in[e,1]^2,\\ \max(x,y) & \text{if } 1>\max(x,y)>e>\min(x,y)>s\\ & \text{or } 1=\max(x,y)>e>\min(x,y)\geq t,\\ \min(x,y) & \text{otherwise},\end{cases}$$

with the restrictions that when S_U is nilpotent it must be $s=t$ and expressions (ii) and (iv) can not be fulfilled.

1.13.5 Case T_U idempotent, S_U continuous

Theorem 1.36 (cf. [156, 222]). *Let U be a uninorm with the neutral element $e\in(0,1)$. If $T_U=T_M$ and S_U is a continuous t-conorm, i.e., it is an ordinal sum of Archimedean t-conorm, then $U(x,y)\in\{x,y\}$ for all $(x,y)\in A(e)$.*

Because of this properties the full characterization will be completed in Section 1.13.7.

1.13.6 Case T_U and S_U Archimedean

This case was already investigated dividing the study in some different cases, when both operators are strict (see [98] and also [150]), when both are nilpotent (see [154] and also [222]), and when one is strict and the other is nilpotent (see [153]), where such uninorms are called weakly continuous uninorms. In these part we characterize all possible uninorms with these underlying operators.

Theorem 1.37 ([90, 155, 154]). *A binary operation $U:[0,1]\to[0,1]$ is a uninorm U such that T_U and S_U are Archimedean if and only if U is given by one of the following possibilities:*

(i) $U \in \mathscr{U}_{\min}$,
(ii) $U \in \mathscr{U}_{\max}$,

(iii) $$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e},\frac{y}{e}\right) & \textit{if } (x,y) \in [0,e]^2, \\ e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \textit{if } (x,y) \in [e,1]^2, \\ 1 & \textit{if } \max(x,y) = 1, \\ \min(x,y) & \textit{otherwise,} \end{cases}$$

(iv) $$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e},\frac{y}{e}\right) & \textit{if } (x,y) \in [0,e]^2, \\ e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \textit{if } (x,y) \in [e,1]^2, \\ 1 & \textit{if } \max(x,y) = 1 \textit{ and } \min(x,y) \neq 0, \\ \min(x,y) & \textit{otherwise,} \end{cases}$$

(v) $$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e},\frac{y}{e}\right) & \textit{if } (x,y) \in [0,e]^2, \\ e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \textit{if } (x,y) \in [e,1]^2, \\ 0 & \textit{if } \min(x,y) = 0, \\ \max(x,y) & \textit{otherwise,} \end{cases}$$

(vi) $$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e},\frac{y}{e}\right) & \textit{if } (x,y) \in [0,e]^2, \\ e+(1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \textit{if } (x,y) \in [e,1]^2, \\ 0 & \textit{if } \min(x,y) = 0 \textit{ and } \max(x,y) \neq 1, \\ \max(x,y) & \textit{otherwise,} \end{cases}$$

(vii) U is a representable uninorm,

with the restriction that expressions $(iv),(v),(vi)$ *and* (vii) *can not be fulfilled when* T_U *is nilpotent, and expressions* $(iii),(iv),(vi)$ *and* (vii) *can not be fulfilled when* S_U *is nilpotent.*

Corollary 1.2. *[154] Let U be a uninorm with the neutral element* $e \in (0,1)$ *such that T_U is nilpotent and S_U is nilpotent. Then either one of the following two statements holds:*

(i) $U \in \mathscr{U}_{\min}$,

(ii) $U \in \mathscr{U}_{\max}$.

1.13.7 Locally internal uninorms in $A(e)$ *with continuous underlying operators*

In all particular cases given in the previous section all uninorms locally internal in $A(e)$ with the given underlying operators have been characterized. Moreover, in the three first cases it is proved that uninorms locally internal in $A(e)$ are in fact all possible ones. On the other hand, in the case when T_U and S_U are Archimedean not all uninorms are locally internal in $A(e)$ because when T_U and S_U are strict, the class of representable uninorms is also possible. However, based on results in [150, 155, 154], again all possible uninorms with the given Archimedean operators are characterized.

In this section we want to present the general case when the underlying t-norm and t-conorm are continuous, but given by ordinal sums. It is clear that in the general case not all possible uninorms will be locally internal in $A(e)$.

Theorem 1.38. *Let $U : [0,1]^2 \to [0,1]$ be a binary operation and $e \in (0,1)$. The following items are equivalent:*

1. *U is a uninorm with the neutral element e, locally internal in $A(e)$ and such that the restrictions of U to the squares $[0,e]^2$ and $[e,1]^2$ are continuous and given by the ordinal sums $(\langle a_i, b_i, T_i\rangle)_{i\in I}$ and $(\langle c_j, d_j, S_j\rangle)_{j\in J}$ respectively.*
2. *There exists decreasing and Id-symmetrical function $g : [0,1] \to [0,1]$ with $g(e) = e$ satisfying the following conditions:*
   - *i) $g(x) \in [e,1] \setminus \bigcup_{j\in J}(c_j, d_j)$ for all $x \in [0,e]$ and, similarly, $g(x) \in [0,e] \setminus \bigcup_{i\in I}(a_i, b_i)$ for all $x \in [e,1]$.*
   - *ii) For all $i \in I$, g is constant in the interval (a_i, b_i), say $g(x) = s_i$ for all $x \in (a_i, b_i)$, and also g is constant in the interval (c_j, d_j) for all $j \in J$, say $g(x) = r_j$ for all $x \in (c_j, d_j)$.*
   - *iii) If T_i is a nilpotent t-norm for some $i \in I$ then $g(a_i) = s_i$ and $U(x, s_i) = \min(x, s_i) = x$ for all $x \in [a_i, b_i)$, or $U(x, s_i) = \max(x, s_i) = s_i$ for all $x \in [a_i, b_i)$. Similarly, if S_j is a nilpotent t-conorm for some $j \in J$ then $g(d_j) = r_j$ and $U(x, r_j) = \max(x, r_j) = x$ for all $x \in (c_j, d_j]$, or $U(x, r_j) = \min(x, r_j) = r_j$ for all $x \in (c_j, d_j]$,*

   such that U is given by the ordinal sum $(\langle a_i, b_i, T_i\rangle)_{i\in I}$ in $[0,e]^2$, by $(\langle c_j, d_j, S_j\rangle)_{j\in J}$ in $[e,1]^2$, and by the expression:

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x) \text{ and } x < g(g(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x) \text{ and } x > g(g(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g(g(x)), \end{cases} \tag{1.29}$$

   in $A(e)$, being commutative on the set of points $(x, g(x))$ such that $x = g^2(x)$.

1.13.8 Uninorms with continuous underlying operators – general case

In this section, we present different ways to characterize uninorms with the continuous underlying t-norm and t-conorm. The first is related to the set-valued function, which is closely related to the set of discontinuity points of the uninorm. The second way is to use the ordinal sum construction in the sense of Clifford. The last one, and in my opinion the most intuitive, is related to the characterization of locally internal uninorms on $A(e)$, and more precisely, it is related to separating functions. However, the choice of how to characterize this type of uninorms depends on where the uninorm is to be applied and which type of description is best suited to the properties being used (see also [241, 160, 243]).

1.13.8.1 Characterization using set-valued function

Mesiarová-Zemánková in papers [202, 203] gave the characterization of uninorms with the continuous underlying t-norm and t-conorm through set-valued functions. We will recall here the necessary definitions and theorems.

Definition 1.18. A mapping $p : [0,1] \to \mathscr{P}([0,1])$ is called a set-valued function on $[0,1]$. Assuming the standard order on $[0,1]$, a set-valued function p is called

(i) non-increasing if for all x_1, $x_2 \in [0,1]$, $x_1 < x_2$, we have $y_1 \geq y_2$ for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$ and thus $p(x_1)$ and $p(x_2)$ intersect in, at most, a single point,
(ii) symmetric if for $x, y \in [0,1]$ there is $y \in p(x)$ if and only if $x \in p(y)$,
(iii) u-surjective if for all $y \in [0,1]$ there exists an $y \in [0,1]$ such that $y \in p(x)$.

$G(p) = \{(x,y) \in [0,1]^2 : y \in p(x)\}$ denotes the graph of a set-valued function p.

Let $A = \inf\{x : U(x,0) > 0\}$, $B = \sup\{x : U(x,1) < 1\}$ then A and B are idempotent elements of U and either $A = 1$, $B \neq 0$, or $A \neq 1$, $B = 0$, or $A = 1$, $B = 0$. Additionally, if U is a uninorm with the continuous underlying t-norm and t-conorm, then there exist idempotent points $a, d \in [0,1]$, $a \leq e \leq d$, such that if $U(x,y) = e$ for some $x, y \in [0,1]$ then $x, y \in (a,d) \cup \{e\}$. Here either $U(x,y) = e$ implies $x = y = e$, in which case $a = d = e$, or otherwise U can attain the value e only on the set $(a,d)^2$. Further, for all $x \in (a,d) \cup \{e\}$ there exists a $y \in (a,d) \cup \{e\}$ such that $U(x,y) = e$. Note that if $a < e$ then U is on $[a,d]^2$ isomorphic to a representable uninorm. We also have

$$0 \leq B \leq a \leq e \leq d \leq A \leq 1.$$

Theorem 1.39 (cf. [203]). *Let U be a uninorm with the continuous underlying t-norm and t-conorm. Then there exists a symmetric, u-surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$ and $U(x,y) = e$ implies $(x,y) \in G(r)$ for all $(x,y) \in [0,1]^2$. Note that U need not to be non-continuous in all points from $G(r)$.*

The set-valued function from Theorem 1.39 will be called the characterizing set-valued function of a uninorm U for uninorm with the continuous underlying t-norm and t-conorm. This function is given by

$$r(x) = \begin{cases} \{1\} & \text{if } x \in (0,B), \\ \{0\} & \text{if } x \in (A,1), \\ [0,B] & \text{if } x = 1, \\ [A,1] & \text{if } x = 0, \\ \{y : U(x,y) = e\} & \text{if } x \in (a,d) \cup \{e\}, \\ \{y : (x,y) \in R^*\} & \text{otherwise}, \end{cases} \tag{1.30}$$

where $R^* = \{(x,y) \in [0,1]^2 : U \text{ is non-continuous in } (x,y)\}$.

Example 1.15. Let U_1 be a representable uninorm with the neutral element $e = 0.5$ such that $U_1(x, 1-x) = 0.5$ for all $x \in (0,1)$ and T a continuous t-norm, S a continuous t-conorm. Then

$$U(x,y) = \begin{cases} \frac{1}{3}T(3x,3y) & \text{if } x,y \in [0,\frac{1}{3}], \\ \frac{1}{3} + \frac{1}{3}U_1(3x-1,3y-1) & \text{if } x,y \in [\frac{1}{3},\frac{2}{3}], \\ \frac{2}{3} + \frac{1}{3}S(3x-2,3y-2) & \text{if } x,y \in [\frac{2}{3},1], \\ \min(x,y) & \text{if } (x,y) \in [0,\frac{1}{3}) \times (\frac{1}{3},\frac{2}{3}) \cup (\frac{1}{3},\frac{2}{3}) \times [0,\frac{1}{3}), \\ \max(x,y) & \text{otherwise} \end{cases}$$

is a uninorm (see Figure 1.15) with the characterizing set valued function

$$r(x) = \begin{cases} [\frac{2}{3},1] & \text{if } x = 0, \\ \{\frac{2}{3}\} & \text{if } x \in (0,\frac{1}{3}), \\ \{1-x\} & \text{if } x \in [\frac{1}{3},\frac{2}{3}], \\ [0,\frac{1}{3}] & \text{if } x = \frac{2}{3}, \\ \{0\} & \text{if } x \in (\frac{2}{3},1]. \end{cases}$$

Note, that the underlying t-norm T_U is an ordinal sum of T and T_{U_1}, the underlying t-conorm S_U is an ordinal sum of S_{U_1} and S. So, both operation are continuous.

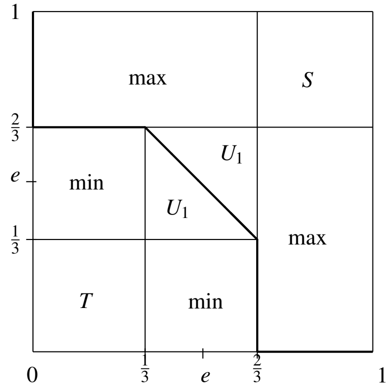

Fig. 1.15 The uninorm from Example 1.15

Remark 1.11. The characterizing set-valued function r divides the uninorm U into two parts: U on points below the characterizing set-valued function attains values less than e, and U on points above the characterizing set-valued function attains values greater than e.

Theorem 1.40 (cf. [203]). *U is a uninorm with the continuous underlying t-norm and t-conorm if and only if there exists a symmetric, u-surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$, and in each point $(x,y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous, or continuous.*

1.13.8.2 Characterization using ordinal sum construction

Definition 1.19 ([200]). Let $a,b,c,d \in [0,1]$ with $a < b < c < d$. Then:

(i) a semigroup $((a,b) \cup \{v\} \cup (c,d), *)$ will be called a representable semigroup if $*$ is isomorphic via (1.19) to a restriction of a representable uninorm on $[0,1]^2$ to $(0,1)^2$,
(ii) a semigroup $((a,b), *)$ will be called a t-strict semigroup if $*$ is linearly isomorphic to a restriction of a strict t-norm on $[0,1]^2$ to $(0,1)^2$,
(iii) a semigroup $((c,d), *)$ will be called an s-strict semigroup if $*$ is linearly isomorphic to a restriction of a strict t-conorm on $[0,1]^2$ to $(0,1)^2$,
(iv) a semigroup $([a,b), *)$ will be called a t-nilpotent semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-norm on $[0,1]^2$ to $[0,1)^2$,
(v) a semigroup $((c,d], *)$ will be called an s-nilpotent semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-conorm on $[0,1]^2$ to $(0,1]^2$,
(vi) a semigroup $((a,b) \cup (c,d), *)$ will be called a d-internal semigroup if $*$ is isomorphic via (1.19) to a restriction of an uninorm locally internal on $A(e)$ on $[0,1]^2$ to $((0,1) \setminus \{e\})^2$,
(vii) a semigroup $((a,b), *)$ will be called a t-internal semigroup if $*$ is linearly isomorphic to the min on $(0,1)^2$,
(viii) a semigroup $((c,d), *)$ will be called an s-internal semigroup if $*$ is linearly isomorphic to the max on $(0,1)^2$.

The next step is to divide the unit interval into appropriate sub-intervals related to the set-valued function.

Definition 1.20. Let U be a uninorm with the continuous underlying t-norm and t-conorm and let $r : [0,1] \to P([0,1])$ be its characterizing set-valued function. Then:

(i) the set $I \subset [0,1]$ is called a maximal horizontal segment of r if $Card(I) > 1$ and there exists a $y \in [0,1]$ such that $y \in p(x)$ if and only if $x \in I$,
(ii) if for $x \in [0,1]$ there is $Card(r(x)) > 1$ then the set $\{x\}$ is called a maximal vertical segment of r,
(iii) the interval $[a,b]$ is called a strictly decreasing segment of r if for all $x \in (a,b)$ we have
$$Card(r(x)) = 1, \;\; Card(r(\max(r(x))) = 1,$$
(iv) the interval $[a,b]$ is called a maximal strictly decreasing segment of r if there is no interval $[c,d]$ which is a strictly decreasing segment of r such that $[a,b] \subsetneq [c,d]$.

The monotonicity of r implies that all maximal segments are intervals. Further, a subinterval of a maximal horizontal segment will be called a horizontal segment. The symmetry of r implies that a maximal horizontal segment I can be paired with a maximal vertical segment $\{y\}$ for which we have $y \in r(x)$ for all $x \in I$. Then $I \times \{y\}$ as well as $\{y\} \times I$ belong to the graph of r.

Lemma 1.8 ([200]). *Let U be a uninorm with the continuous underlying t-norm and t-conorm and let $r : [0,1] \to \mathscr{P}([0,1])$ be its characterizing set-valued function. Then all maximal segments of r are closed intervals.*

Let denote by S_r the set of end points of all maximal segments of r and by $\overline{S}_r$ its closure. Note that there is a countable number of maximal horizontal and strictly decreasing segments and due to the symmetry of r there is also a countable number of maximal vertical segments. Therefore $\overline{S}_r$ is countable. Then we have the following result.

Theorem 1.41 ([200]). *Let U be a uninorm with the continuous underlying t-norm and t-conorm, let $r : [0,1] \to \mathscr{P}([0,1])$ be its characterizing set-valued function and assume $x \in [0,1]$. Then either $x \in \overline{S}_r$ or x is an interior point of exactly one maximal segment of r. Moreover end points of all types of maximal segments of r are idempotent points.*

Remark 1.12. Let U be a uninorm such that the underlying operations are Archimedean t-norm and t-conorm respectively. Then either its characterizing set-valued function is strictly decreasing on $[0,1]$ (in this case it is a representable uninorm), or the interval $[0,e]$ ($[e,1]$) is a horizontal segment of r (in this case there is $y \in [0,1]$ such that $r(x) = \{y\}$ for all $x \in (0,e)$ or all $x \in (e,1)$). For the value y we then have $y \in \{1,e\}$ or $y \in \{0,e\}$.

Lemma 1.9 ([200]). *Let U be a uninorm with the continuous underlying t-norm and t-conorm and let r be its characterizing set-valued function. Then if $a,b \in [0,1]$, $a < b$, are idempotent elements of U such that there is no idempotent element in (a,b) and there exists $x \in (a,b)$ such that $U(x,x) = a$ ($U(x,x) = b$) then r on $[a,b]$ corresponds to a horizontal segment.*

The next results described connections between maximal strictly decreasing segments.

Lemma 1.10 (cf. [200]). *Let U be a uninorm with the continuous underlying t-norm and t-conorm and let r be its characterizing set-valued function. The following conditions are fulfilled:*

(i) If $a,b \in [0,1]$, $a < b \le e$, are idempotent elements of U such that there is no idempotent element in (a,b) and r on $[a,b]$ corresponds to a strictly decreasing segment then $d = \min(r(a))$ and $c = \max(r(b))$ are idempotent elements of U such that there is no idempotent element in (c,d) and r on $[c,d]$ corresponds to a strictly decreasing segment. Further, $a = \max(r(\min(r(a))))$ and $b = \min(r(\max(r(b))))$.

(ii) If $c,d \in [0,1]$, $e \le c < d$, are idempotent elements of U such that there is no idempotent element in (c,d) and r on $[c,d]$ corresponds to a strictly decreasing segment then $b = \min(r(c))$ and $a = \max(r(d))$ are idempotent elements of U such that there is no idempotent element in (a,b) and r on $[a,b]$ corresponds to a strictly decreasing segment. Further, $c = \max(r(\min(r(c))))$ and $d = \min(r(\max(r(d))))$.

(iii) If $a,b \in [0,1]$, $a < b \le e$, are such that $U(x,x) = x$ for all $x \in [a,b]$ and r on $[a,b]$ corresponds to a strictly decreasing segment then for $d = \min(r(a))$ and $c = \max(r(b))$ we have $U(y,y) = y$ for all $y \in [c,d]$ and r on $[c,d]$ corresponds to a strictly decreasing segment. Further, $a = \max(r(\min(r(a))))$ and $b = \min(r(\max(r(b))))$.

(iv) If $c,d \in [0,1]$, $e \le c < d$, are such that $U(y,y) = y$ for all $y \in [c,d]$ and r on $[c,d]$ corresponds to a strictly decreasing segment then for $b = \min(r(c))$ and $a = \max(r(d))$ we have $U(x,x) = x$ for all $x \in [a,b]$ and r on $[a,b]$ corresponds to a strictly decreasing segment. Further, $c = \max(r(\min(r(c))))$ and $d = \min(r(\max(r(d))))$.

Definition 1.21 ([200]). Let us define the function $u : [0,1] \to [0,1]$ as follows $u(x) = U(x,x)$. Then $I_U = \{x \in [0,1] : u(x) = x\}$ is a closed set and $[0,e] \setminus I_U = \bigcup_{k \in K}(a_k, b_k)$, where $\{(a_k,b_k)\}_{k \in K}$ is a family of a countable number of open and disjoint subintervals of $[0,e]$ for some index set K. Similarly, $[e,1] \setminus I_U = \bigcup_{l \in L}(c_l, d_l)$, where $\{(c_l,d_l)\}_{l \in L}$ is a family of a countable number of open and disjoint subintervals of $[e,1]$ for some index set L such that $K \cap L = \emptyset$.

Now let $K_1 = \{k \in K : a_k = U(x,x)$ for some $x \in [0,1], x \neq a_k\}$ and $K_2 = \{k \in K :$ r on $[a_k,b_k]$ corresponds to a strictly decreasing segment$\}$. Then $K_1 \cap K_2 = \emptyset$. Let $K_3 = K \setminus (K_1 \cup K_2)$. Similarly, let $L_1 = \{l \in L : d_k = U(x,x)$ for some $x \in [0,1], x \neq d_k\}$ and $L_2 = \{l \in L : r$ on $[c_l,d_l]$ corresponds to a strictly decreasing segment$\}$ and $L_3 = L \setminus (L_1 \cup L_2)$. Then $L_1 \cup L_2 = \emptyset$. Due to Lemma 1.10 each $k \in K_2$ can be paired with some $l \in L_2$ and vice-versa. So we can use $l \in L_2$ and the corresponding $k \in K_2$ interchangeably.

Denote $X = \{x \in [0,1] : x$ is an end point of a maximal segment of $r\}$ and $B = \bigcup_{k \in K}(a_k,b_k) \cup \bigcup_{k \in K_1}\{a_k\} \cup \bigcup_{k \in K_2}\{U(b_k,c_k)\}$. Let $[0,e] \setminus (B \cup X) = \bigcup_{m \in M} Y_m$, where the sets Y_m are components of $[0,e] \setminus (B \cup X)$ with respect to connectedness. Note that we can select such an M that K, L, M are mutually disjoint. Additionally, denote $A^* = \{supY_m, \inf Y_m : m \in M\} \setminus (B \cup \{e\})$ and $Z_m = Y_m \setminus A^*$ for all $m \in M$, $M_* = \{m \in M : Z_m \neq \emptyset\}$, $M_1 = \{m \in M_* : r$ on (a_m,b_m) corresponds to a horizontal segment$\}$, $M_2 = \{m \in M_* : r$ on (a_m,b_m) corresponds to a strictly decreasing segment$\}$, $A = [0,e) \setminus (B \cup \bigcup_{m \in M_*} Z_m)$. Let $i : A \to M_3$ be an isomorphism, where M_3 is a index set disjoint with all previous index sets, $Z_m = \{i(m)\}$ for all $m \in M_3$.

Similarly, let $C = \bigcup_{l \in L}(c_l,d_l) \cup \bigcup_{l \in L_1}\{d_k\} \cup \bigcup_{k \in K_2}\{U(b_k,c_k)\}$, $[e,1] \setminus (C \cup X) = \bigcup_{o \in O} Y_o$, where the sets Y_o are components of $[e,1] \setminus (C \cup X)$ with respect to connectedness, $D^* = \{\sup Y_o, \inf Y_o : o \in O\} \setminus (C \cup \{e\})$, $Z_o = Y_o \setminus D$ for all $o \in O$, $O_* = \{o \in O : Z_o \neq \emptyset\}$, $O_1 = \{o \in O_* : r$ on (c_o,d_o) corresponds to a horizontal segment $\}$, $O_2 = \{o \in O_* : r$ on (c_o,d_o) corresponds to a strictly decreasing segment $\}$, $D = (e,1] \setminus (C \cup \bigcup_{o \in O_*} Z_o)$. Let $j : D \to O_3$ be an isomorphism, where O_3 is a index set disjoint with all previous index sets, $Z_o = \{j(o)\}$ for all $o \in O_3$

Due to Lemma 1.10 each $m \in M_2$ can be paired with some $o \in O_2$ and vice-versa. So we can use $o \in O_2$ and the corresponding $m \in M_2$ interchangeably. If $U(x,y) = e$ for some $x \neq e$ then the point e is already covered in the used partition, however, if $U(x,y) = e$ implies $x = y = e$ then we should add a separate set $\{e\}$.

Therefore we have a partition of $[0,e]$ into sets: $[a_k,b_k)$ for $k \in K_1$, $(a_k,b_k) \cup (\{U(b_k,c_k)\} \cap [0,e])$ for $k \in K_2$, (a_k,b_k) for $k \in K_3$, (a_m,b_m) for $m \in M_1 \cup M_2$, $\{a_m\}$ for $m \in M_3$ and eventually $\{e\}$.

Similarly, we have a partition of $[e,1]$ into sets: $(c_l,d_l]$ for $l \in L_1$, $(c_k,d_k) \cup (\{U(b_k,c_k)\} \cap [e,1])$ for $k \in K_2$, (c_l,d_l) for $l \in L_3$, (c_o,d_o) for $o \in O_1 \cup M_2$, $\{d_o\}$ for $o \in O_3$ and eventually $\{e\}$.

Denote $P^* = K_1 \cup K_2 \cup K_3 \cup M_1 \cup M_2 \cup M_3 \cup L_1 \cup L_3 \cup O_1 \cup O_3$. If $U(x,y) = e$ implies $x = y = e$ then we additionally assume an index $p^* \notin P^*$ and the set $X_{p^*} = \{e\}$. Then $P = P^* \cup \{p^*\}$. In the other case we put $P = P^*$. For $p \in P$ we will denote corresponding sets described above by X_p. Obviously $X_{p_1} \cap X_{p_2} = \emptyset$ for $p_1, p_2 \in P$, $p_1 \neq p_2$.

In Theorem 1.42 we assume to apply the partition from Definition 1.21.

Theorem 1.42 ([200]). *U is a uninorm with the continuous underlying t-norm and t-conorm if and only if it is an ordinal sum of the following semigroups:*

(i) a t-nilpotent semigroup on $[a_k,b_k)$ for all $k \in K_1$,
(ii) a representable semigroup on $(a_k,b_k) \cup \{U(b_k,c_k)\} \cup (c_k,d_k)$ for all $k \in K_2$,
(iii) a t-strict semigroup on (a_k,b_k) for all $k \in K_3$,
(iv) an s-nilpotent semigroup on $(c_l,d_l]$ for all $l \in L_1$,
(v) an s-strict semigroup on (c_l,d_l) for all $l \in L_3$,
(vi) a t-internal semigroup on (a_m,b_m) for all $m \in M_1$,
(vii) a d-internal semigroup on $(a_m,b_m) \cup (c_m,d_m)$ for all $m \in M_2$,
(viii) a semigroup defined on $\{a_m\}$ for all $m \in M_3$,
(ix) an s-internal semigroup on (c_o,d_o) for all $o \in O_1$,
(x) a semigroup defined on $\{d_o\}$ for all $o \in O_3$,
(xi) eventually a semigroup defined on $\{e\}$.

1.13.8.3 Characterization using separating functions

For locally internal uninorms there exists a separating function g_U such that above this function the uninorm is equal to the maximum, and below it is equal to the minimum. Similarly, if we assume that the uninorm is locally internal on $A(e)$, we get the function g_U with analogous properties. At the same time, the properties of this function affect the form of the uninorm in the rest of the domain (see Theorem 1.27). If we drop the assumption that U is locally internal on $A(e)$, we can define two functions (using the Lemma 1.4):

$$\underline{g_U}(x) = \sup\{y : U(x,y) = \min(x,y)\} \tag{1.31}$$

$$\overline{g_U}(x) = \inf\{y : U(x,y) = \max(x,y)\} \tag{1.32}$$

Note that using the fact that e is the neutral element of the uninorm U and monotonicity of U the functions $\underline{g_U}$, $\overline{g_U}$ are well defined, and we have the following dependencies

Lemma 1.11. *Let U be a uninorm and $\underline{g_U}$, $\overline{g_U}$ be defined by* (1.31) *and* (1.32). *Then*

$$\underline{g_U}(e) = \overline{g_U}(e) = e,$$

$$e \leq \underline{g_U}(x) \leq \overline{g_U}(x) \leq 1 \ \text{for } x \in [0,e),$$

$$0 \leq \underline{g_U}(x) \leq \overline{g_U}(x) \leq e \ \text{for } x \in (e,1].$$

Moreover both these functions are decreasing.

Lemma 1.12 ([82]). *Let U be a uninorm with the continuous underlying t-norm and t-conorm. Then for each idempotent element $a \in [0,1]$ of U uninorm U is internal on $\{a\} \times [0,1]$.*

Remark 1.13. Let I denote the set of idempotent elements of a uninorm U with the continuous underlying t-norm and t-conorm. Then $U(x,y) \in \{\min(x,y), \max(x,y)\}$ for all $(x,y) \in I \times [0,1] \cup [0,1] \times I$, i.e. $\underline{g_U}(x) = \overline{g_U}(x)$ on I.

In the rest of this subsection we will use the notations adopted in the Definition 1.21. Using Theorem 1.42 we can generalize Remark 1.13

Lemma 1.13. *Let U be a uninorm with the continuous underlying t-norm and t-conorm. Then for each $p \in P$ uninorm U is locally internal on $X_p \times ([0,1] \setminus X_p)$, where X_p is one of the sets listed in the Theorem 1.42.*

Proof. This fact follows directly from Theorem 1.42 and 1.15. □

Remark 1.14. Note that only for $p \in K_2$ we do not get a locally internal operation on $(X_p \cap [0,e]) \times [e,1]$ and $(X_p \cap [e,1]) \times [0,e]$. More precisely, on the set $((a_p,b_p) \cup \{U(b_p,c_p)\} \cup (c_p,d_p))^2 \cap A(e)$ uninorm is not locally internal. On the other hand, uninorm is locally internal on the set $([0,e] \setminus \bigcup_{p \in K_2} X_p) \times [e,1]$ and $([e,1] \setminus \bigcup_{p \in K_2} X_p) \times [0,e]$, and consequently $g(x) := \underline{g_U}(x) = \overline{g_U}(x)$ for all $x \in [0,1] \setminus \bigcup_{p \in K_2}$.

Lemma 1.14. *Let U be a uninorm with the continuous underlying t-norm and t-conorm and let r be its characterizing set-valued function. Then $g(x) = \max(r(x))$ for all $x \in [0,1] \setminus \bigcup_{k \in K_2} ((a_k,b_k) \cup (c_k,d_k))$ such that $card(r(x)) = 1$ and moreover $g(x) \in [\min(r(x)), \max(r(x))]$ for all $x \in [0,1] \setminus \bigcup_{k \in K_2} ((a_k,b_k) \cup (c_k,d_k))$ such that $card(r(x)) > 1$.*

Proof. Observe, that on the set $[0,1] \setminus \bigcup_{k \in K_2} ((a_k,b_k) \cup (c_k,d_k))$ the completed graph of g coincides with the graph of set-valued function r, because the only point of discontinuity of the locally internal uninorm on this set may be the point that separates min from max. So we get the thesis. □

Lemma 1.15. *Let U be a uninorm with the continuous underlying t-norm and t-conorm and $k \in K_2$. If $x \in (a_k, b_k)$ then $U(x,y) = \min(x,y)$ for all $y \in [e, c_k]$ and $U(x,y) = \max(x,y)$ for all $y \in [d_k, 1]$. If $x \in (c_k, d_k)$ then $U(x,y) = \min(x,y)$ for all $y \in [0, a_k]$ and $U(x,y) = \max(x,y)$ for all $y \in [d_k, e]$.*

Proof. Suppose $x \in (a_k, b_k)$ for some $k \in K_2$. According to Lemma 1.13 $U(x,y) \in \{x,y\}$ for $y \in [e,1] \setminus (c_k, d_k)$. According to the definition of the set K_2, the set-valued function takes the value $r(x) \subset (c_k, d_k)$ and according to Remark 1.11 for $y > \max(r(x))$ we have $U(x,y) > e$, and for $y < \min(r(x))$ we have $U(x,y) < e$. So $U(x,y) = \min(x,y)$ for all $y \in [e, c_k]$ and $U(x,y) = \max(x,y)$ for all $y \in [d_k, 1]$. For $x \in (c_k, d_k)$ the proof is analogous. □

Lemma 1.16. *Let U be a uninorm with the continuous underlying t-norm and t-conorm, and $k \in K_2$. If $x \in (a_k, b_k)$ then $\underline{g_U}(x) = c_k$ and $\overline{g_U}(x) = d_k$. If $x \in (c_k, d_k)$ then $\underline{g_U}(x) = a_k$ and $\overline{g_U}(x) = b_k$.*

Proof. Suppose $x \in (a_k, b_k)$ for some $k \in K_2$. According to Lemma 1.15 $U(x,y) = \max(x,y)$ for $y \in [d_k, 1]$. Taking $y \in (c_k, d_k)$ and using the fact that U on $(a_k, b_k) \cup \{U(b_k, c_k)\} \cup (c_k, d_k)$ is isomorphic to representable semigroup we obtain inequality $x < U(x,y) < y$. So, $d_k = \inf\{y \in [e,1] : U(x,y) = \max(x,y)\}$ and consequently $\overline{g_U}(x) = d_k$. Similarly, $c_k = \sup\{y \in [e,1] : U(x,y) = \min(x,y)\} = \underline{g_U}(x)$. For $x \in (c_k, d_k)$ the proof is analogous. □

Remark 1.15. Note that the case where $\underline{g_U}(x) \neq \overline{g_U}(x)$ can occur if $\min(x, \min r(x)) < U(x, \min r(x)) < \max(x, \min r(x))$, where r is the multi-valued function given by (1.30). This is the case *(*ii) in Theorem 1.42.

Lemma 1.17. *Let U be a uninorm with the continuous underlying t-norm and t-conorm. Then the functions $\underline{g_U}$ and $\overline{g_U}$ given by* (1.31) *and* (1.32)*, respectively are* Id-symmetrical.

Proof. Let r be characterizing set-valued function. Then by Lemma 1.14 on the set $[0,1] \setminus \bigcup_{k \in K_2} ((a_k, b_k) \cup (c_k, d_k))$ the completed graph of g coincides with the graph of set-valued function r and in this set $g = \underline{g_U} = \overline{g_U}$. Moreover, completed graph of g in this set is *Id*-symmetrical. Furthermore, by Lemma 1.16 $\overline{g_U}(x) = d_k$ for all $x \in (a_k, b_k)$ and by Theorem 1.42 point d_k is a point of discontinuity of $\overline{g_U}$ ($\overline{g_U}(d_k) \leq a_k$) and we obtain a symmetry of the completed graph for interval connected with K_2. Similarly we can prove, that completed graph of $\underline{g_U}$ is *Id*-symmetrical. □

Theorem 1.43. *Let $U : [0,1] \to [0,1]$ be a binary operation and $e \in (0,1)$. The following items are equivalent:*

1. *U is a uninorm with the neutral element e and continuous underlying t-norm and t-conorm (i.e. the restrictions of U to the squares $[0,e]^2$ and $[e,1]^2$ are the ordinal sums $(\langle a_k, b_k, T_k \rangle)_{k \in K}$ and $(\langle c_l, d_l, S_l \rangle)_{l \in L}$ respectively of Archimedean components).*
2. *There exist decreasing and Id-symmetrical functions $\underline{g} : [0,1] \to [0,1]$ and $\overline{g} : [0,1] \to [0,1]$ with $\underline{g}(e) = \overline{g}(e) = e$ satisfying the following conditions:*

i) $\underline{g}(x), \overline{g}(x) \in [e,1] \setminus \bigcup_{l \in L} (c_l, d_l)$ *for all* $x \in [0,e]$ *and, similarly,* $\underline{g}(x), \overline{g}(x) \in [0,e] \setminus \bigcup_{k \in K} (a_k, b_k)$ *for all* $x \in [e,1]$.

ii) For all $k \in K$, $\underline{g}$ *and* $\overline{g}$ *are constant in the interval* (a_k, b_k), *say* $\underline{g}(x) = s_k$ *for all* $x \in (a_k, b_k)$, $\overline{g}(x) = t_k$ *for all* $x \in (a_k, b_k)$, *and also* $\underline{g}$ *and* $\overline{g}$ *are constant in the interval* (c_l, d_l) *for all* $l \in L$, *say* $\underline{g}(x) = s_l$ *for all* $x \in (c_l, d_l)$, $\overline{g}(x) = t_l$ *for all* $x \in (c_l, d_l)$.

iii) $\underline{g}(x) = \overline{g}(x)$ *for all* $x \in [0,1] \setminus \left(\bigcup_{k \in K_2} (a_k, b_k) \cup \bigcup_{k \in K_2} (c_k, d_k)\right)$,

iv) If T_k *is a nilpotent t-norm for some* $k \in K_1$ *then* $s_k = t_k$, $\underline{g}(a_k) = \overline{g}(a_k) = s_k$ *and* $U(x, s_k) = \min(x, s_k) = x$ *for all* $x \in [a_k, b_k)$, *or* $U(x, s_k) = \max(x, s_k) = s_k$ *for all* $x \in [a_k, b_k)$. *Similarly, if* S_l *is a nilpotent t-conorm for some* $l \in L_1$ *then* $s_l = t_l$, $\underline{g}(d_l) = \overline{g}(d_l) = s_l$ *and* $U(x, s_l) = \max(x, s_l) = x$ *for all* $x \in (c_l, d_l]$, *or* $U(x, s_l) = \min(x, s_l) = s_l$ *for all* $x \in (c_l, d_l]$,

v) For $k \in K_2$ *we have* $\underline{g}(x) < \overline{g}(x)$ *for all* $x \in (a_k, b_k)$, $c_k = s_k$ *and* $d_k = t_k$, *and* $\underline{g}(x) < \overline{g}(x)$ *for all* $x \in (c_k, d_k)$. *Moreover,* $U|_{((a_k,b_k) \cup \{U(b_k,c_k)\} \cup (c_k,d_k))^2}$ *is isomorphic with representable uninorm* U_k,

such that U *is given by the ordinal sum* $(\langle a_k, b_k, T_k \rangle)_{k \in K}$ *in* $[0,e]^2$, *by* $(\langle c_l, d_l, S_l \rangle)_{l \in L}$ *in* $[e,1]^2$, *and by the expression:*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < \underline{g}(x) \text{ or } y = \underline{g}(x) \text{ and } x < \underline{g}(\underline{g}(x)) \\ \max(x,y) & \text{if } y > \overline{g}(x) \text{ or } y = \overline{g}(x) \text{ and } x > \overline{g}(\overline{g}(x)) \\ U_{\nu_k}^{a_k,b_k,c_k,d_k}(x,y) & \text{if } x \in (a_k,b_k) \cup (c_k,d_k),\ k \in K_2,\ \underline{g_U}(x) < y < \overline{g_U}(x) \\ x \text{ or } y & \text{if } y = \underline{g}(x) \text{ and } x = \underline{g}(\underline{g}(x)), \\ & \text{or } y = \overline{g}(x) \text{ and } x = \overline{g}(\overline{g}(x)) \end{cases} \tag{1.33}$$

in $A(e)$, *being commutative on the set of points* $(x, \underline{g}(x))$ *such that* $x = \underline{g}^2(x)$ *and* $(x, \overline{g}(x))$ *such that* $x = \overline{g}^2(x)$, *where* $U_{\nu_k}^{a_k,b_k,c_k,d_k}$ *is given by* (1.19) *and* $\nu_k = U(b_k, c_k)$.

Proof. Let us prove first $1 \Rightarrow 2$.
By Lemmas 1.13–1.17 functions $\underline{g} : [0,1] \to [0,1]$ and $\overline{g} : [0,1] \to [0,1]$ are decreasing and Id-symmetrical. Moreover $\underline{g}(e) = \overline{g}(e) = e$. Condition i) follows from Remarks 1.13 and 1.14 and the fact that representable semigroup on $(a_k, b_k) \cup \{U(b_k, c_k)\} \cup (c_k, d_k)$ for all $k \in K_2$ are summand of the uninorm U (see Theorem 1.42). If $k \in K_2$, then item ii) follows from Lemma 1.16. If $k \in K \setminus K_2$, then by definition of K_2 we have, that r on (a_k, b_k) corresponds to horizontal segment. By Lemma 1.14 we obtain, that $\underline{g} = \overline{g}$ are constant. Item iii) follows directly by Remark 1.14. Items iv) and v) and form of U follow directly from the representation of ordinal sum of appropriate semigroup (compare Theorem 1.42).

Conversely, suppose that U is given by the corresponding ordinal sums on $[0,e]$ and $[e,1]$ and by expression (1.33) in $A(e)$, being commutative on the set of points $(x, g(x))$ such that $x = g^2(x)$. We want to prove that U is a uninorm with the neutral element e. Clearly, U is increasing in each place and has the neutral element e. Let us define the operations $\widetilde{U}_{\min}, \widetilde{U}_{\max} : [0,1]^2 \to [0,1]$ as follows

$$\widetilde{U}_{\min}(x,y) = \begin{cases} \min(x,y) & \text{if } x \in (a_k,b_k) \cup (c_k,d_k),\ k \in K_2,\ \underline{g_U}(x) < y < \overline{g_U}(x), \\ U(x,y) & \text{otherwise.} \end{cases}$$

$$\widetilde{U}_{\max}(x,y) = \begin{cases} \max(x,y) & \text{if } x \in (a_k,b_k) \cup (c_k,d_k),\ k \in K_2,\ \underline{g_U}(x) < y < \overline{g_U}(x), \\ U(x,y) & \text{otherwise.} \end{cases}$$

Both functions $\widetilde{U}_{\min}$, $\widetilde{U}_{\max}$ are locally internal on $A(e)$, with associated functions $\overline{g_U}$ and $\underline{g_U}$. By Theorem 1.38 $\widetilde{U}_{\min}$, $\widetilde{U}_{\max}$ are uninorms with neutral element e. Furthermore, $U(x,y) = \widetilde{U}_{\min}(x,y) = \widetilde{U}_{\max}(x,y)$ for all points (x,y) in the set $[0,1]^2 \setminus D$, where

$$D = \left(\bigcup_{i \in K_2} (a_i,b_i) \times (c_i,d_i) \right) \cup \left(\bigcup_{i \in K_2} (c_i,d_i) \times (a_i,b_i) \right).$$

Directly from this fact we obtain that U is commutative on the set $[0,1]^2 \setminus D$ beyond the points $(x, g(x))$ such that $x = g^2(x)$ and consequently U is commutative.

Moreover, to prove the associativity of U we only need to check the equality

$$U(U(x,y),z) = U(x,U(y,z)) \tag{1.34}$$

on sets $\{x,y,z\}$ such that at least one of the pairs in the set $\{x,y,z\} \times \{x,y,z\}$ belongs to D. On the rest of the domain the associativity of U follows directly from the associativity of $\widetilde{U}_{\min}$.

Suppose that $x,y \in (a_i,b_i)$, $z \in (c_j,d_j)$ for some $i,j \in K_2$.

Suppose that $x \in (a_i,b_i)$, $y \in (c_j,d_j)$ for some $i,j \in K_2$. Thus, $\underline{g}((a_i,b_i)) = \{c_i\}$, $\overline{g}((a_i,b_i)) = \{d_i\}$. Moreover, $U(x,y) \in (a_k,b_k) \cup \{U(b_k,c_k)\} \cap [0,e]$ or $U(x,y) \in (c_k,d_k) \cup \{U(b_k,c_k)\} \cap [e,1]$. Using the commutativity of U we need to consider the following cases:

a) If $z \le a_i$ or $z \ge d_i$, then
$U(U(x,y),z) = z = U(x,U(y,z))$,
$U(U(x,z),y) = z = U(x,U(z,y))$,
$U(U(z,x),y) = z = U(z,U(x,y))$.
$U(U(z,y),x) = z = U(z,U(y,x))$,
$U(U(y,x),z) = z = U(y,U(x,z))$,
$U(U(y,z),x) = z = U(y,U(z,x))$,

b) If $z \in (a_i,b_i) \cup \{U(b_i,c_i)\} \cup (c_i,d_i)$, then associativity follows from the associativity of the operation $U_{v_k}^{a_k,b_k,c_k,d_k}$,

c) If $z \in ([b_i,e) \cup (e,c_i]) \setminus \{U(b_i,c_i)\}$, then
$U(U(x,y),z) = U(x,y) = U(x,U(y,z))$,

Therefore, U is associative. □

Remark 1.16. In Theorem 1.43 functions $\underline{g}$ and $\overline{g}$ are strictly related to the multi-valued function from Section 1.13.8.1, where the segments for which the multi-function is decreasing are divided into two groups:

1. When its graph separates min from max – here the range is disjoint from $\bigcup(a_i,b_i)$. Then $\underline{g} = \overline{g}$ and elements in this range are idempotent

2. When a given segment coincides with one of the ranges (a_i, b_i) or (c_i, d_i). Then U on $(a_i, b_i) \cup (c_i, d_i)$ is isomorphic to a representable uninorm.

1.14 Uninorms locally internal on the boundary

In this section we present the description of the uninorms which are locally internal on the boundary, i.e. on the set $[0,1]^2 \setminus (0,1)^2$. We will denote the family of all such uninorms by $\mathscr{U}_{locb}$. As we can see, all the uninorms discussed in the previous sections are uninorms locally internal on the boundary.

Lemma 1.18 ([155, 183]). *Let $U : [0,1]^2 \to [0,1]$ be a uninorm with the neutral element $e \in (0,1)$. Then the following statements hold:*

(i) If T_U is continuous then $U(1,y) \in \{y,1\}$ for all $y \in [0,1]$,
(ii) If S_U is continuous then $U(0,y) \in \{0,y\}$ for all $y \in [0,1]$.

Remark 1.17. The uninorms from the class $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ need not have continuous underlying t-norms and t-conorms, but are locally internal on the boundary.

Directly from the monotonicity of the uninorm we get

Lemma 1.19. *Let $U : [0,1]^2 \to [0,1]$ be a uninorm with the neutral element $e \in (0,1)$. If U is locally internal on the boundary then:*

- *There exist an element $a \in [0,e]$ such that $U(x,1) = x$ for all $x < a$ and $U(x,1) = 1$ for all $x > a$.*
- *There exist an element $b \in [e,1]$ such that $U(x,0) = 0$ for all $x < b$ and $U(x,0) = x$ for all $x > b$.*

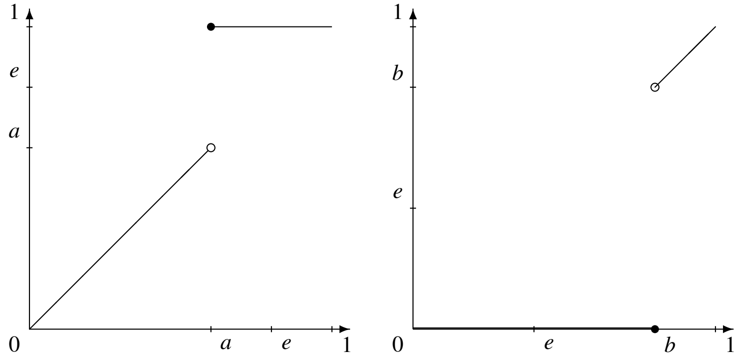

Fig. 1.16 Boundary functions $U(x,1)$

Remark 1.18. Note that for conjunctive uninorms $b = 1$, and for disjunctive uninorms $a = 0$.
Moreover, for conjunctive uninorms one of the following cases may occur: $U(a,1) = a$ or $U(a,1) = 1$ (see Figure 1.16 left part). Similarly, for disjunctive uninorms one of the following cases can occur: $U(b,0) = 0$ or $U(b,0) = b$ (see Figure 1.16 right part).

Considering the above and the results from the paper [151] we obtain the results in the case where $U(a,1) = 1$.

Theorem 1.44. *Operation $U : [0,1]^2 \to [0,1]$ is a conjunctive uninorm with the neutral element e and there exist an element $a \in [0,e]$ such that $U(x,1) = x$ for all $x < a$ and $U(x,1) = 1$ for all $x \geq a$, if and only if T_U is ordinal sum of two t-norm $\big(\langle 0, \frac{a}{e}, T_1\rangle, \langle \frac{a}{e}, 1, T_2\rangle\big)$, $U|_{[a,1]^2}$ is isomorphic to disjunctive uninorm and*

$$U(x,y) = \begin{cases} aT_1(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a)^2, \\ a + (e-a)T_2(\frac{x-a}{e-a}, \frac{y-a}{e-a}) & \text{if } (x,y) \in [a,e]^2, \\ \min(x,y) & \text{if } (x,y) \in [0,a) \times [a,1] \cup [a,1] \times [0,a), \\ \max(x,y) & \text{if } (x,y) \in [a,e) \times \{1\} \cup \{1\} \times [a,e), \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2. \end{cases} \tag{1.35}$$

Proof. This theorem follows directly from Lemmas 2–5 and Theorem 3 in [151]. □

Example 1.16. Operation $U : [0,1]^2 \to [0,1]$ is a conjunctive uninorm, locally internal on the boundary with the neutral element $e = \frac{1}{2}$, where

$$U(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0, \frac{1}{4})^2, \\ \frac{1}{4} & \text{if } (x,y) \in [\frac{1}{4}, \frac{1}{2})^2, \\ \max(x,y) & \text{if } (x,y) \in [\frac{1}{4}, \frac{1}{2}) \times \{1\} \cup \{1\} \times [\frac{1}{4}, \frac{1}{2}) \cup [\frac{1}{2}, 1]^2. \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Remark 1.19. Note that if in Theorem 1.44 $a = e$, then we get a uninorm from the class $\mathcal{U}_{\min}$.

Similarly, if $U(a,1) = a$ we have

Theorem 1.45. *Operation $U : [0,1]^2 \to [0,1]$ is a conjunctive uninorm with the neutral element e and there exist an element $a \in [0,e]$ such that $U(x,1) = x$ for all $x \leq a$ and $U(x,1) = 1$ for all $x > a$, if and only if T_U is an ordinal sum of t-subnorm and positive t-norm $\big(\langle 0, \frac{a}{e}, M_1\rangle, \langle \frac{a}{e}, 1, T_2\rangle\big)$, $U|_{(a,1]^2}$ is isomorphic to a conjunctive uninorm and*

$$U(x,y)=\begin{cases} aM_1(\frac{x}{a},\frac{y}{a}) & \textit{if } (x,y)\in[0,a]^2,\\ a+(e-a)T_2(\frac{x-a}{e-a},\frac{y-a}{e-a}) & \textit{if } (x,y)\in(a,e]^2,\\ \min(x,y) & \textit{if } (x,y)\in[0,a]\times(a,1]\cup(a,1]\times[0,a],\\ \max(x,y) & \textit{if } (x,y)\in(a,e)\times\{1\}\cup\{1\}\times(a,e),\\ e+(1-e)S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \textit{if } (x,y)\in[e,1]^2. \end{cases} \tag{1.36}$$

Proof. This theorem follows directly from Lemmas 6–7 and Theorem 4 in [151]. □

Remark 1.20. Note that if in Theorem 1.45 $a=0$ then T_U must be a positive t-norm.

By duality we can obtain the results for disjunctive uninorms.

Theorem 1.46. *Operation $U:[0,1]^2\to[0,1]$ is a disjunctive uninorm with the neutral element e and there exist an element $b\in[e,1]$ such that $U(x,0)=0$ for all $x\le b$ and $U(x,0)=x$ for all $x>b$, if and only if S_U is an ordinal sum of two t-conorm $\big(\langle 0,\frac{b}{e},S_1\rangle,\langle\frac{b}{e},1,S_2\rangle\big)$, $U|_{[0,b]^2}$ is isomorphic to a conjunctive uninorm and*

$$U(x,y)=\begin{cases} eT_U(\frac{x}{e},\frac{y}{e}) & \textit{if } (x,y)\in[0,e]^2,\\ e+(b-e)S_1(\frac{x-e}{b-e},\frac{y-e}{b-e}) & \textit{if } (x,y)\in[e,b]^2,\\ b+(1-b)S_2(\frac{x-b}{1-b},\frac{y-b}{1-b}) & \textit{if } (x,y)\in(b,1]^2,\\ \max(x,y) & \textit{if } (x,y)\in(b,1]\times[0,b]\cup[0,b]\times(b,1],\\ \min(x,y) & \textit{if } (x,y)\in(e,b]\times\{0\}\cup\{0\}\times(e,b]. \end{cases} \tag{1.37}$$

Similarly, if $U(b,0)=b$ we have

Theorem 1.47. *Operation $U:[0,1]^2\to[0,1]$ is a disjunctive uninorm with the neutral element e and there exist an element $b\in[e,1]$ such that $U(x,0)=0$ for all $x<b$ and $U(x,0)=x$ for all $x\ge b$, if and only if S_U is an ordinal sum of positive t-conorm and t-superconorm $\big(\langle 0,\frac{b-e}{1-e},S_1\rangle,\langle\frac{b-e}{1-e},1,S_2\rangle\big)$, $U|_{[0,b)^2}$ is isomorphic to a disjunctive uninorm and*

$$U(x,y)=\begin{cases} eT_U(\frac{x}{e},\frac{y}{e}) & \textit{if } (x,y)\in[0,e]^2,\\ e+(b-e)S_1(\frac{x-e}{b-e},\frac{y-e}{b-e}) & \textit{if } (x,y)\in[e,b)^2,\\ b+(1-b)S_2(\frac{x-b}{1-b},\frac{y-b}{1-b}) & \textit{if } (x,y)\in[b,1]^2,\\ \max(x,y) & \textit{if } (x,y)\in[b,1]\times[0,b)\cup[0,b)\times[b,1],\\ \min(x,y) & \textit{if } (x,y)\in(e,b)\times\{0\}\cup\{0\}\times(e,b). \end{cases} \tag{1.38}$$

1.15 Uninorms not locally internal on the boundary

In this section, we will present the structure of uninorms that do not belong to any of the previously discussed classes, i.e. those that are not locally internal on the boundary. We will denote the family of all such uninorms by $\mathscr{U}_{nlocb}$. Some information about this kind of uninorm we can find in [119, 121, 155, 223].

Let denote by $f_1(x) = U(x,1)$, $f_0(x) = U(0,x)$ (see Section 1.7)

Lemma 1.20 (cf. [119, 155, 5]). *Let $U : [0,1]^2 \to [0,1]$ be a uninorm with the neutral element $e \in (0,1)$. Then, for every $x \in (0,1)$, $U(x,1) = x$ or $U(x,1) = x_0 > x$. Furthermore, if $U(x,1) = x_0 > x$ then $U(z,1) = x_0$ for all $z \in [x,x_0]$.*
Let $s = \inf\{x : U(x,1) = x_0\}$. Then $s \in (0,1)$ is a point of discontinuity of the function f_1.

Lemma 1.21 (cf. [119, 155, 5]). *Let $U : [0,1]^2 \to [0,1]$ be a uninorm with the neutral element $e \in (0,1)$. Then, for every $x \in (0,1)$, $U(0,x) = x$ or $U(0,x) = x_0 < x$. Furthermore, if $U(0,x) = x_0 < x$ then $U(0,z) = x_0$ for all $z \in [x_0,x]$.*
Let $s = \sup\{x : U(0,x) = x_0\}$. Then $s \in (0,1)$ is a point of discontinuity of the function f_0.

Remark 1.21. Each conjunctive uninorm U is discontinuous at point $a = \inf\{x \in [0,1] : U(x,1) = 1\}$. If a conjunctive uninorm U is not locally internal on the boundary then there exists $x \in [0,a)$ such that $x_0 = U(x,1) \notin \{x,1\}$, which means that $x < x_0 < 1$. Then from Lemma 1.20 we know that U is discontinuous at point $b = \inf\{x \in [0,1] : U(x,1) = x_0\}$. Similar result we can obtain for disjunctive uninorms. Therefore one of the boundary functions of a uninorm which is not locally internal on the boundary (f_1 or f_0) has at least two points of discontinuity.

Taking into account the above remark, we can consider two or more points of discontinuity. Each point of discontinuity is associated with a segment $(b,x_0]$. Later in this section we will consider the case when the function f_1 has exactly two discontinuity points and one of them is 0 or e.

1.15.1 Disjunctive case

If a uninorm is disjunctive, the function f_1 is a constant function and $f_1(x) = 1$ for each $x \in [0,1]$ and the uninorm in this part of the boundary is locally internal. So we only need to consider the function f_0

If 1 is one of the points of discontinuity of the function f_0, then it takes one of the forms

Lemma 1.22 ([260]). *Let U be a conjunctive uninorm with the neutral element $e \in (0,1)$. Then the boundary function f_0 of U is discontinuous only at points 1 and $a \in [e,1)$ if and only if*

$$f_0(x) = \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in [b,1), \\ x & \text{if } x \in (a,b), \\ 0 & \text{otherwise}, \end{cases} \tag{1.39}$$

or

$$f_0(x) = \begin{cases} 1 & \text{if } x = 1, \\ b & \text{if } x \in [b,1), \\ x & \text{if } x \in [a,b), \\ 0 & \text{otherwise}, \end{cases} \tag{1.40}$$

for some $b \in [a,1)$.

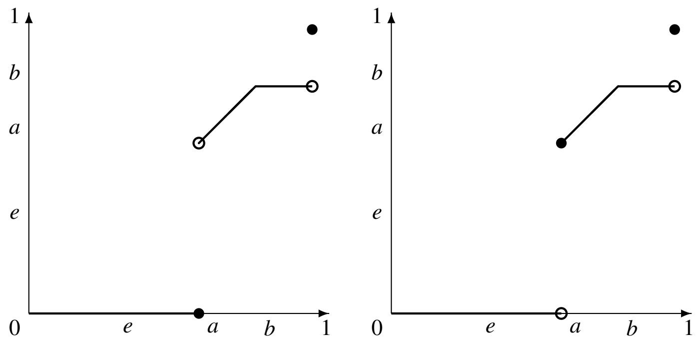

Fig. 1.17 Functions f_0 given by (1.39) (left) and f_0 given by (1.40) (right)

Theorem 1.48 (cf. [260]). *Let* $U : [0,1]^2 \to [0,1]$ *be a binary operation and* $0 < e \leq a < b < 1$. *Then U is a disjunctive uninorm with the neutral element e and the boundary function* f_0 *given by* (1.39) *if and only if there exist a t-conorm* S_0, *conjunctive uninorm* U_0 *with the neutral element* $\frac{e}{a}$, *increasing function* $B : [0,a] \times [b,1) \to [b,1)$ *such that*

$$U(x,y) = \begin{cases} a + (1-a)S_0(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in (a,1]^2, \\ aU_0(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a]^2, \\ B(x,y) & \text{if } (x,y) \in [0,a] \times [b,1), \\ B(y,x) & \text{if } (x,y) \in [b,1) \times [0,a], \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{1.41}$$

and

- $S_0(x,y) = 1$ *for* $(x,y) \in (0,1] \times [\frac{b-a}{1-a}, 1] \cup [\frac{b-a}{1-a}, 1) \times (0,1]$,
- $S_0(x,y) \in \{1\} \cup (0, \frac{b-a}{1-a}]$ *for* $(x,y) \in (0, \frac{b-a}{1-a})^2$,
- $B(0,y) = b$ *and* $B(e,y) = y$ *for* $y \in [b,1)$, $B(x,b) = b$ *for* $x \in [0,a]$,
- $B(x, B(y,z)) = B\left(aU_0\left(\frac{x}{a}, \frac{y}{a}\right), z\right)$ *for* $x,y \in [0,a]$ *and* $z \in [b,1)$.

Theorem 1.49 (cf. [260]). *Let* $U : [0,1]^2 \to [0,1]$ *be a binary operation and* $0 < e < a \leq b < 1$. *Then U is a disjunctive uninorm with the neutral element e and*

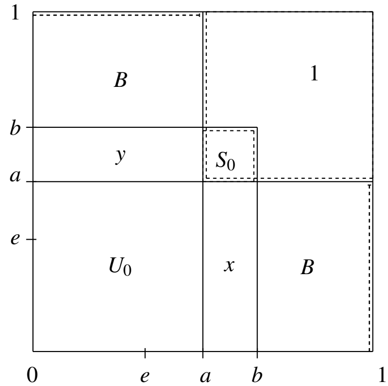

Fig. 1.18 The uninorm given by (1.41)

the boundary function f_0 *given by* (1.40) *if and only if there exist t-superconorm* S_0*, disjunctive uninorm* U_0 *with the neutral element* $\frac{e}{a}$ *and the positive underlying t-conorm* S_{U_0}*, increasing function* $B : [0,a) \times [b,1) \to [b,1)$ *such that*

$$U(x,y) = \begin{cases} a+(1-a)S_0(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in [a,1]^2, \\ aU_0(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a)^2, \\ B(x,y) & \text{if } (x,y) \in [0,a) \times [b,1), \\ B(y,x) & \text{if } (x,y) \in [b,1) \times [0,a), \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{1.42}$$

and

- $S_0(x,y) = 1$ *for* $(x,y) \in [0,1] \times [\frac{b-a}{1-a}, 1] \cup [\frac{b-a}{1-a}, 1] \times [0,1]$,
- $S_0(x,y) \in \{1\} \cup [0, \frac{b-a}{1-a}]$ *for* $(x,y) \in [0, \frac{b-a}{1-a})^2$,
- $B(0,y) = b$ *and* $B(e,y) = y$ *for* $y \in [b,1)$*,* $B(x,b) = b$ *for* $x \in [0,a)$*,*
- $B(x, B(y,z)) = B\left(aU_0\left(\frac{x}{a}, \frac{y}{a}\right), z\right)$ *for* $x,y \in [0,a)$ *and* $z \in [b,1)$.

Example 1.17. Let

$$U(x,y) = \begin{cases} 0 & \text{if } \min(x,y) = 0, \ \max(x,y) \le 0.5, \\ \frac{2xy}{4xy+(1-2x)(1-2y)} & \text{if } (x,y) \in (0,0.5]^2, \\ \max(x,y) & \text{if } 0.5 < \max(x,y) < 0.75, \\ 0.75 & \text{if } (x,y) \in [0,0.5] \times [0.75,1) \cup [0.75,1) \times [0,0.5], \\ 1 & \text{otherwise}. \end{cases}$$

U is a uninorm with the neutral element $e = 0.25$, which is not locally internal on the boundary, because $U(0,0.9) = 0,75$.

If e is one of the points of discontinuity of the function f_0 then the following property holds.

Lemma 1.23 ([260]). *Let U be a conjunctive uninorm with the neutral element $e \in (0,1)$. Then the boundary function f_0 of U is discontinuous only at points e and $b \in (e,1]$ if and only if*

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0,e], \\ a & \text{if } x \in [a,b], \\ x & \text{otherwise}, \end{cases} \tag{1.43}$$

or

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0,e], \\ a & \text{if } x \in [a,b), \\ x & \text{otherwise}, \end{cases} \tag{1.44}$$

for some $a \in (e,b)$.

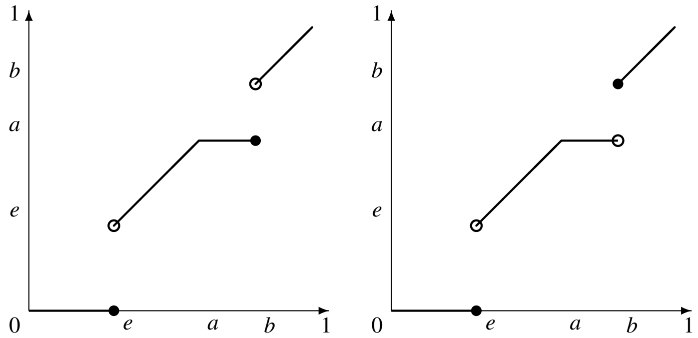

Fig. 1.19 Functions f_0 given by (1.43) (left) and f_0 given by (1.44) (right)

Theorem 1.50 (cf. [260]). *Let $U : [0,1]^2 \to [0,1]$ be a binary operation and $0 < a < b < e < 1$. Then U is a disjunctive uninorm with the neutral element e and the boundary function f_0 given by (1.43) if and only if there exist a t-norm T, t-conorm S and increasing function $B : [0,e) \times [a,b] \to [a,b]$ such that*

$$U(x,y) = \begin{cases} eT(x/e, y/e) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ B(x,y) & \text{if } (x,y) \in [0,e) \times [a,b], \\ B(y,x) & \text{if } (x,y) \in [a,b] \times [0,e), \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{1.45}$$

and

- $S(\frac{a-e}{1-e}, y) = S(\frac{b-e}{1-e}, y) > \frac{b-e}{1-e}$ *for* $y > 0$,
- $S(x,y) \in (\frac{b-e}{1-e}, 1) \cup (0, \frac{a-e}{1-e})$ *for* $(x,y) \in (0, \frac{a-e}{1-e})^2$,
- $B(x,y) \in [a,y] \subset [a,b]$, $B(0,y) = a$ *for* $y \in [a,b]$ *and* $B(x,a) = a$ *for* $x \in [0,e)$,

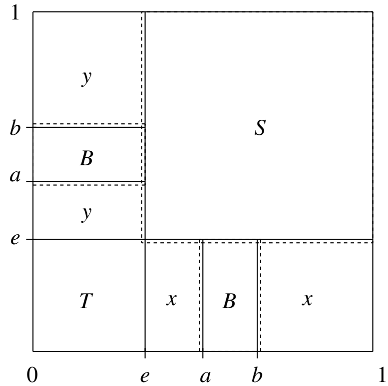

Fig. 1.20 The uninorm given by (1.45)

- $B(x,B(y,z)) = B\left(eT\left(\frac{x}{e},\frac{y}{e}\right),z\right)$ *for* $x,y \in [0,e)$ *and* $z \in [a,b]$.

Theorem 1.51 (cf. [260]). *Let* $U : [0,1]^2 \to [0,1]$ *be a binary operation and* $0 < a < b < e \le 1$. *Then* U *is a disjunctive uninorm with the neutral element* e *and the boundary function* f_0 *given by* (1.44) *if and only if there exist a t-norm* T, *t-conorm* S *and increasing function* $B : [0,e) \times [a,b) \to [a,b)$ *such that*

$$U(x,y) = \begin{cases} eT(x/e,y/e) & \text{if } (x,y) \in [0,e]^2, \\ e+(1-e)S(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ B(x,y) & \text{if } (x,y) \in [0,e) \times (a,b], \\ B(y,x) & \text{if } (x,y) \in (a,b] \times [0,e), \\ \max(x,y) & \text{otherwise}, \end{cases} \tag{1.46}$$

and

- $S(x,y) = S(\frac{a-e}{1-e},y) \ge \frac{b-e}{1-e}$ *for* $(x,y) \in (0,1] \times [\frac{a-e}{1-e},\frac{b-e}{1-e})$,
- $S(x,y) \in [\frac{b-e}{1-e},1) \cup (0,\frac{a-e}{1-e}]$ *for* $(x,y) \in (0,\frac{a-e}{1-e})^2$,
- $B(x,y) \in [a,y] \subset [a,b)$, $B(0,y) = a$ *for* $y \in [a,b)$ *and* $B(x,a) = a$ *for* $x \in [0,e)$,
- $B(x,B(y,z)) = B\left(eT\left(\frac{x}{e},\frac{y}{e}\right),z\right)$ *for* $x,y \in [0,e)$ *and* $z \in [a,b)$.

1.15.2 Conjunctive case

If a uninorm is conjunctive, the function f_0 is a constant function and $f_0(x) = 0$ for each $x \in [0,1]$ and the uninorm in this part of the boundary is locally internal. So we only need to consider the function f_1. Therefore we get dual considerations in relation to the disjunctive case.

If 0 is one of the points of discontinuity of the function f_1, then it takes one of the forms given in the following lemma.

Lemma 1.24 ([260]). *Let U be a conjunctive uninorm with the neutral element $e \in (0,1)$. Then the boundary function f_1 of U is non-continuous only at points 0 and $b \in (0,e]$ if and only if*

$$f_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x \in (0,a], \\ x & \text{if } x \in (a,b), \\ 1 & \text{otherwise}, \end{cases} \tag{1.47}$$

or

$$f_1(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x \in (0,a], \\ x & \text{if } x \in (a,b], \\ 1 & \text{otherwise}, \end{cases} \tag{1.48}$$

for some $a \in (0,b]$.

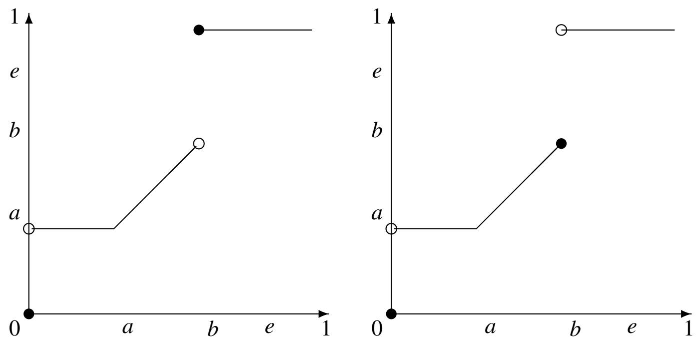

Fig. 1.21 Functions f_1 given by (1.47) (left) and f_1 given by (1.48) (right)

Theorem 1.52 (cf. [260]). *Let $U : [0,1]^2 \to [0,1]$ be a binary operation and $0 < a < b \leq e < 1$. Then U is a conjunctive uninorm with the neutral element e and the boundary function f_1 given by (1.47) if and only if there exist a t-norm T_0, a disjunctive uninorm U_0 with the neutral element $\frac{e-b}{1-b}$, increasing function $A_0 : (0,a] \times [b,1] \to (0,a]$ such that*

$$U(x,y) = \begin{cases} bT_0(x/b, y/b) & \text{if } (x,y) \in [0,b)^2, \\ b + (1-b)U_0(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \text{if } (x,y) \in [b,1]^2, \\ A_0(x,y) & \text{if } (x,y) \in (0,a] \times [b,1], \\ A_0(y,x) & \text{if } (x,y) \in [b,1] \times (0,a], \\ \min(x,y) & \text{otherwise}, \end{cases} \tag{1.49}$$

and

- $T_0(x,y) = 0$ *for* $(x,y) \in [0,a/b] \times [0,1) \cup [0,1) \times [0,a/b]$,

- $T_0(x,y) \in \{0\} \cup [a/b,1)$ *for* $(x,y) \in (a/b,1)^2$,
- $A_0(x,1) = a$ *for* $x \in (0,a]$, $A_0(a,y) = a$ *for* $y \in [b,1]$ *and* $A_0(x,e) = x$ *for* $x \in (0,a]$,
- $A_0\left(x, b+(1-b)U_0\left(\frac{y-b}{1-b}, \frac{z-b}{1-b}\right)\right) = A_0(A_0(x,y),z)$ *for* $x \in (0,a]$ *and* $y,z \in [b,1]$.

Theorem 1.53 (cf. [260]). *Let* $U : [0,1]^2 \to [0,1]$ *be a binary operation and* $0 < a \leq b < e < 1$. *Then* U *is a conjunctive uninorm with the neutral element* e *and the boundary function* f_1 *given by* (1.48) *if and only if there exist a t-subnorm* T_0, *conjunctive uninorm* U_0 *with the neutral element* $\frac{e-b}{1-b}$ *and positive underlying t-norm* T_{U_0}, *increasing function* $A_0 : (0,a] \times (b,1] \to (0,a]$ *such that*

$$U(x,y) = \begin{cases} bT_0(x/b, y/b) & \textit{if } (x,y) \in [0,b]^2, \\ b+(1-b)U_0(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \textit{if } (x,y) \in (b,1]^2, \\ A_0(x,y) & \textit{if } (x,y) \in (0,a] \times (b,1], \\ A_0(y,x) & \textit{if } (x,y) \in (b,1] \times (0,a], \\ \min(x,y) & \textit{otherwise}, \end{cases} \tag{1.50}$$

and

- $T_0(x,y) = 0$ *for* $(x,y) \in [0,a/b] \times [0,1] \cup [0,1] \times [0,a/b]$,
- $T_0(x,y) \in \{0\} \cup [a/b,1]$ *for* $(x,y) \in (a/b,1]^2$,
- $A_0(x,1) = a$ *for* $x \in (0,a]$, $A_0(a,y) = a$ *for* $y \in (b,1]$ *and* $A_0(x,e) = x$ *for* $x \in (0,a]$,
- $A_0\left(x, b+(1-b)U_0\left(\frac{y-b}{1-b}, \frac{z-b}{1-b}\right)\right) = A_0(A_0(x,y),z)$ *for* $x \in (0,a]$ *and* $y,z \in (b,1]$.

If e is one of the points of discontinuity of the function f_1 then we have the following property.

Lemma 1.25 ([260]). *Let* U *be a conjunctive uninorm with the neutral element* $e \in (0,1)$. *Then the boundary function* f_1 *of* U *is non-continuous only at points* e *and* $a \in [0,e)$ *if and only if*

$$f_1(x) = \begin{cases} 1 & \textit{if } x \in [e,1], \\ b & \textit{if } x \in [a,b), \\ x & \textit{otherwise}, \end{cases} \tag{1.51}$$

or

$$f_1(x) = \begin{cases} 1 & \textit{if } x \in [e,1], \\ b & \textit{if } x \in (a,b], \\ x & \textit{otherwise}, \end{cases} \tag{1.52}$$

for some $b \in (a,e)$.

Theorem 1.54 (cf. [260]). *Let* $U : [0,1]^2 \to [0,1]$ *be a binary operation and* $0 < a < b < e < 1$. *Then* U *is a conjunctive uninorm with the neutral element* e *and the*

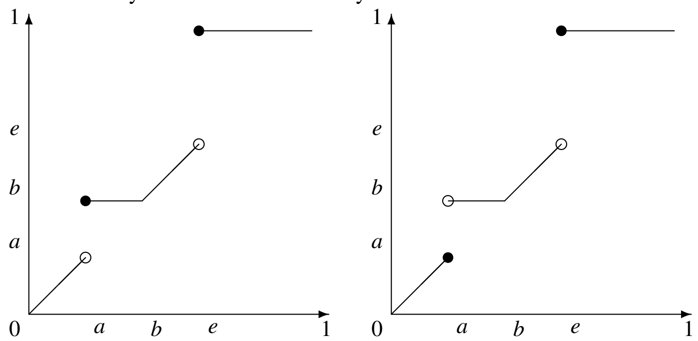

Fig. 1.22 Functions f_1 given by (1.51) (left) and f_1 given by (1.52) (right)

boundary function f_1 given by (1.51) *if and only if there exist t-norm T, t-conorm S and increasing function $A_2 : [a,b] \times (e,1] \to [a,b]$ such that*

$$U(x,y) = \begin{cases} eT_0(x/e,y/e) & \text{if } (x,y) \in [0,e]^2, \\ e+(1-e)S(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ A_2(x,y) & \text{if } (x,y) \in [a,b] \times (e,1], \\ A_2(y,x) & \text{if } (x,y) \in (e,1] \times [a,b], \\ \min(x,y) & \text{otherwise}, \end{cases} \tag{1.53}$$

and

- $T(x,a/e) = T(x,b/e) < a/e$ *for* $x \in [0,1)$,
- $T(x,y) \in [0,a/e) \cup [b/e,1)$ *for* $(x,y) \in (b/e,1)^2$,
- $A_2(x,y) \in [x,b] \subset [a,b]$, $A_2(x,1) = b$ *for* $x \in [a,b]$ *and* $A_2(b,y) = b$ *for* $y \in (e,1]$,
- $A_2\left(x, e+(1-e)S\left(\frac{y-e}{1-e},\frac{z-e}{1-e}\right)\right) = A_2(A_2(x,y),z)$ *for* $x \in [a,b]$ *and* $y,z \in (e,1]$.

Theorem 1.55 (cf. [260]). *Let $U : [0,1]^2 \to [0,1]$ be a binary operation and $0 \leq a < b < e < 1$. Then U is a conjunctive uninorm with the neutral element e and the boundary function f_1 given by* (1.52) *if and only if there exist t-norm T, t-conorm S and increasing function $A_2 : (a,b] \times (e,1] \to (a,b]$ such that*

$$U(x,y) = \begin{cases} eT(x/e,y/e) & \text{if } (x,y) \in [0,e]^2, \\ e+(1-e)S(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ A_2(x,y) & \text{if } (x,y) \in (a,b] \times (e,1], \\ A_2(y,x) & \text{if } (x,y) \in (e,1] \times (a,b], \\ \min(x,y) & \text{otherwise}, \end{cases} \tag{1.54}$$

and

- $T(x,y) = T(x,b/e) < a/e$ *for* $(x,y) \in [0,1) \times (a/e,b/e]$,
- $T(x,y) \in [0,a/e] \cup [b/e,1)$ *for* $(x,y) \in (b/e,1)^2$,
- $A_2(x,y) \in [x,b] \subset (a,b]$, $A_2(x,1) = b$ *for* $x \in (a,b]$ *and* $A_2(b,y) = b$ *for* $y \in (e,1]$,

- $A_2\left(x, e+(1-e)S\left(\frac{y-e}{1-e}, \frac{z-e}{1-e}\right)\right) = A_2(A_2(x,y),z)$ *for* $x \in (a,b]$ *and* $y,z \in (e,1]$.

1.15.3 General case

Let $|a,b]$ denote one of the intervals $(a,b]$ or $[a,b]$ and $[a,b|$ one of the intervals $[a,b)$ or $[a,b]$. Taking into account Lemma 1.20 for the conjunctive uninorm we can see that the following property holds.

Theorem 1.56. *Let U be a conjunctive uninorm with the neutral element $e \in (0,1)$. Then there exists a countable family $\{|a_k,b_k]\}_{k\in\mathscr{K}}$ of nonoverlapping, proper subintervals of $(0,1]$ such that there exists $k_0 \in \mathscr{K}$ for which $[e,1] \subset |a_{k_0}, b_{k_0}]$ and the boundary function f_1 of U is given by the formula*

$$f_1(x) = \begin{cases} b_k & \text{if } x \in |a_k,b_k], \\ x & \text{otherwise}. \end{cases} \tag{1.55}$$

Remark 1.22. Note that in the above theorem $a_k = \inf\{x \in [0,1] : U(x,1) = b_k\}$ for all $k \in \mathscr{K}$. Moreover, every point a_k for $k \in \mathscr{K}$ is a point of discontinuity of the function f_1.

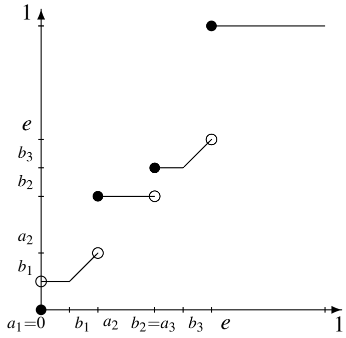

Fig. 1.23 Functions f_1 given by (1.55)

If $card\mathscr{K} = 1$ then we get a uninorms that are locally internal on the boundary. Examples of uninorms from this class are the uninorms of the class $\mathscr{U}_{\min}$, as well as uninorms continuous inside a unit square.

If $card\mathscr{K} = 2$ then we get a uninorms that were described in the Subsection 1.15.2.

Taking into account the Lemma 1.21 for the disjunctive uninorm we can see that the following property holds.

Theorem 1.57. *Let U be a disjunctive uninorm with the neutral element $e \in (0,1)$. Then there exists a countable family $\{[a_k, b_k|\}_{k\in\mathscr{K}}$ of nonoverlapping, proper subintervals of $[0,1)$ such that there exists $k_0 \in \mathscr{K}$ for which $[0,e] \subset [a_{k_0}, b_{k_0}|$ and the boundary function f_0 of U is given by the formula*

$$f_0(x) = \begin{cases} a_k & \text{if } x \in [a_k, b_k|, \\ x & \text{otherwise}. \end{cases} \tag{1.56}$$

Remark 1.23. Note that in the above theorem $b_k = \sup\{x \in [0,1] : U(x,0) = a_k\}$ for all $k \in \mathscr{K}$. Moreover, every point b_k for $k \in \mathscr{K}$ is a point of discontinuity of the function f_0.

If $card\mathscr{K} = 1$ then we get a uninorms that are locally internal on the boundary. Examples of uninorms from this class are the uninorms of the class $\mathscr{U}_{\max}$, as well as uninorms continuous inside a unit square.

If $card\mathscr{K} = 2$ then we get a uninorms that were described in Subsection 1.15.1.

Open Problem 1 *Characterize uninorms that are not locally internal on the boundary:*

1. *for which the number of points of discontinuity is finite,*
2. *for which the number of points of discontinuity is infinite but countable.*

1.16 Relationships between particular classes of uninorms

Let us recall the designations of the described classes of uninorms:

- $\mathscr{U}$ – the family of all uninorms,
- $\mathscr{U}_{\min}$ – the family of uninorms with continuous function $f_1(x) = U(x,1)$ except perhaps at the point $x = e$,
- $\mathscr{U}_{\max}$ – the family of uninorms with continuous function $f_0(x) = U(x,0)$ except perhaps at the point $x = e$,
- $\mathscr{U}_{rep}$ – the family of representable uninorms,
- $\mathscr{U}_{cos}$ – uninorms continuous on $(0,1)^2$,
- $\mathscr{U}_{id}$ – idempotent uninorms,
- $\mathscr{U}_{cuo}$ – uninorms with continuous underlying operations,
- $\mathscr{U}_{loc}$ – locally internal uninorms,
- $\mathscr{U}_{locA}$ – uninorms locally internal on $A(e)$,
- $\mathscr{U}_{locb}$ – uninorms locally internal on the boundary,
- $\mathscr{U}_{nlocb}$ – uninorms not locally internal on the boundary.

In the rest of this Section we will present the relationships between the listed classes of uninorms. Some of them are (proper) subclasses of other classes, while others are disjoint. They can, for example, be described by elements of other classes, as is the case with uninorms not locally internal on the boundary.

$$\mathscr{U}_{rep} \subsetneq \mathscr{U}_{cos} \tag{1.57}$$

Using Theorem 1.24 and 1.25 we see, that every representable uninorm is continuous on the open unit square, but the inverse inclusion does not hold, as shown in the following example.

Example 1.18. The uninorm U given by

$$U(x,y) = \begin{cases} 2xy & \text{if } x,y \in [0,\frac{1}{2}], \\ \frac{1}{2} + \frac{(2x-1)(2y-1)}{6(1-x)(1-y)+(2x-1)(2y-1)} & \text{if } x,y \in [\frac{1}{2},1], \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a uninorm continuous in the open unit square, but not representable.

$$\mathscr{U}_{cos} \subsetneq \mathscr{U}_{cuo} \tag{1.58}$$

Using Lemma 1.5 we see, that every uninorm continuous on the open unit square has the continuous underlying t-norm and t-conorm, but the inverse inclusion does not hold, as shown in Example 1.15, where e.g. $(0,\frac{1}{3}) \times \{\frac{2}{3}\}$ is the set of discontinuity points.

$$\mathscr{U}_{id} = \mathscr{U}_{loc} \tag{1.59}$$

Using the Theorem 1.26 we get the above equality.

$$\mathscr{U}_{loc} \subsetneq \mathscr{U}_{locA} \tag{1.60}$$

Using the Theorem 1.38 we get the above inclusion. The case $i)$ in Theorem 1.35 gives us an example of uninorm locally internal on $A(e)$ but not idempotent.

$$\mathscr{U}_{\min} \subsetneq \mathscr{U}_{locA} \tag{1.61}$$

The uninorm form the class $\mathscr{U}_{\min}$ in the set $A(e)$ is equal to min, i.e. it is locally internal on the boundary.

$$\mathscr{U}_{\max} \subsetneq \mathscr{U}_{locA} \tag{1.62}$$

The uninorm form the class $\mathscr{U}_{\max}$ in the set $A(e)$ is equal to max, i.e. it is locally internal on the boundary.

$$\mathscr{U}_{\min} \cap \mathscr{U}_{cos} = \emptyset \tag{1.63}$$

The uninorms form the class $\mathscr{U}_{\min}$ are discontinuous on the set $\{e\} \times (e,1)$, wheres uninorms from the class $\mathscr{U}_{cos}$ are continuous on this set, i.e. these two classes are disjoint.

$$\mathscr{U}_{\max} \cap \mathscr{U}_{cos} = \emptyset \tag{1.64}$$

The uninorms form the class $\mathscr{U}_{\max}$ are discontinuous on the set $(0,e) \times \{e\}$, wheres uninorms from the class $\mathscr{U}_{cos}$ are continuous on this set, i.e. these two classes are disjoint.

$$\mathscr{U}_{\min} \cap \mathscr{U}_{rep} = \emptyset \text{ and } \mathscr{U}_{\max} \cap \mathscr{U}_{rep} = \emptyset \tag{1.65}$$

This follows directly form the above dependencies and the fact, that $\mathscr{U}_{rep} \subsetneq \mathscr{U}_{cos}$

$$\mathscr{U}_{\min} \cap \mathscr{U}_{cuo} \neq \emptyset \tag{1.66}$$

$$\mathscr{U}_{\max} \cap \mathscr{U}_{cuo} \neq \emptyset \tag{1.67}$$

From Theorem 1.13 we obtain that the underlying t-norm and t-conorm of the uninorm from class $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ can be arbitrary, including continuous one, which gives the above dependencies. Additionally, the following example shows no inclusion between classes.

Example 1.19. Let U be given by

$$U(x,y) = \begin{cases} 0 & \text{if } x,y \in [0,\frac{1}{2}), \\ 1 & \text{if } x,y \in (\frac{1}{2},1], \\ \min(x,y) & \text{otherwise.} \end{cases}$$

This uninorm is in the class $\mathscr{U}_{\min}$, but not in $\mathscr{U}_{cuo}$.
Uninorm $\overline{U_e}$ from Example 1.9 is in the class $\mathscr{U}_{\max}$, but not in $\mathscr{U}_{cuo}$.
The representable uninorms are examples of uninorms with the continuous underlying operators, but not locally internal on $A(e)$.

Lemma 1.26.

$$\mathscr{U} = \mathscr{U}_{locb} \cup \mathscr{U}_{nlocb} \tag{1.68}$$

All relationships presented in this section are included in Figure 1.24.

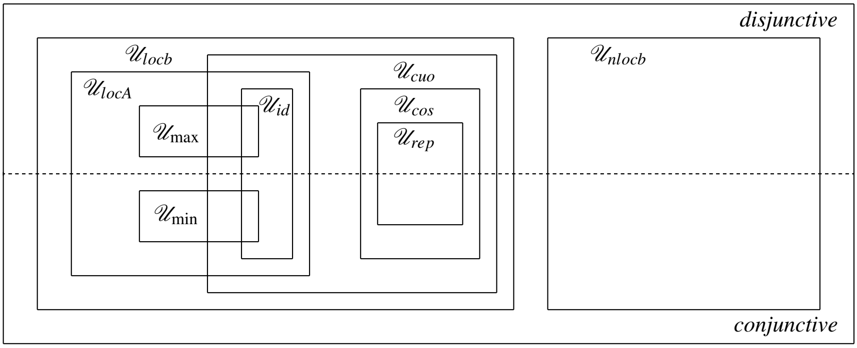

Fig. 1.24 The intersection of different class of uninorms

Chapter 2
Uninorms on the lattice

As complexity rises, precise statements lose meaning and meaningful statements lose precision.

L.A. Zadeh

In fuzzy set theory, uninorms can be used to define new classes of fuzzy sum operators and intersection operators. The generalization of these classes of operators to a bounded poset is very important for defining the union and intersection operators for L-fuzzy sets introduced by Goguen [106].

In 2014 Bodjanova and Kalina [30] introduced uninorms on bounded lattices. They present method of constructing uninorms on bounded lattices. They also introduced uninorms on bounded lattices derived from both given underlying t-norm and t-conorm. Then Karaçal and Mesiar [137] showed the existence of a uninorm with the neutral element e for any element $e \in L \setminus \{0,1\}$ with basic t-norms and t-conorms on an arbitrarily bounded lattice. They also introduced the least and greatest uninorms with the neutral element $e \in L \setminus \{0,1\}$. In 2016 Çaylı, Karaçal and Mesiar [46] showed the existence of idempotent uninorms on bounded lattice for any neutral element $e \in L \setminus \{0,1\}$. Using these construction methods, they obtained the least and greatest idempotent uninorms. In 2018 Çaylı [39] introduced new methods for the construction of uninorms on bounded lattices with the neutral element, based on the existence of t-norms and t-conorms. Differences between these and earlier constructions were also assessed and illustrated. Next, in 2019 [40] she presented two methods of constructing uninorms from given t-norms and t-conorms on bounded lattices under certain constraints and in [41] she studied idempotent uninorms on bounded lattices. She showed that it does not always exist an idempotent uninorm on a bounded lattice that differs from the least and greatest idempotent uninorm. She also proposed two new methods of obtaining uninorms on bounded lattices. Subsequently, Xie and Li [262] proposed two new methods for constructing uninorms on bounded lattices. Then, Dan, Hu and Qiao [55] presented several new constructions of uninorms with additional constraints on the neutral element. They also obtained two classes of idempotent uninorms on bounded lattices. In 2020, Dan and Hu [54] proposed two methods for the construction of uninorms with underlying t-norms and t-conorms. They also gave some examples illustrating the differences between the new construction of uninorms proposed by them and some existing uninorms. Next, Aşıcı and Mesiar [12] constructed a uninorm on an arbitrary bounded lattice L with some constraint by using the knowledge of the existence of t-norms on

bounded lattices. By using this construction method, they obtained idempotent uninorm. In particular, they showed that these new construction methods differ from some existing methods for constructing uninorms on bounded lattices. In the same year, Zhang et al. [269] presented the construction of the class $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ in the lattice using the results of [131], where t-subnorms and t- superconorms are used (see also [125]). Moreover, in the literature we can find constructions of uninorms using closure and interior operators [42, 126, 216], separating functions [43, 44], additive generators [117] or uninorms on sublattices [263].
Special types of lattices are lattices of intervals and finite chains, to which the above-mentioned constructions can be applied. However, in these lattices the description of some uninorm classes may be more detailed. In Section 2.5 we will present uninorms on a lattice of intervals (cf. [69, 68, 80]) and in Section 2.6 we will present uninorms on finite chains (cf. [96, 62, 234, 217]).

2.1 Bounded lattices

Definition 2.1 (cf. [29]). An ordered set L is called a lattice if

$$\forall x,y \in L \;\; x \wedge y \in L, \;\; x \vee y \in L,$$

where $x \wedge y = \inf\{x,y\}$, $x \vee y = \sup\{x,y\}$.

Lattice L is bounded if there exist $0 \in L$ and $1 \in L$ such that for all $x \in L$ we have $0 \leq x \leq 1$.

Example 2.1. $(\mathbb{R}, \leq)$ is a lattice with the operations $x \wedge y = \min\{x,y\}$, $x \vee y = \max\{x,y\}$ for $x,y \in \mathbb{R}$, but it is not bounded.
$([0,1], \leq)$ and set $\{0,1,a,b\}$, with the order presented in the Hasse diagram are bounded lattices (see Figure 2.1).

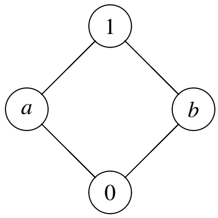

Fig. 2.1 The Hasse diagram with the order in the lattice $\{0,1,a,b\}$

Definition 2.2 ([29]). A lattice $(L, \leq)$ is distributive if the following two equivalent conditions hold:
i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x,y,z \in L$,
ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x,y,z \in L$.

Definition 2.3 ([29]). Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$. If $a \leq b$then we define lattice intervals:

$$[a,b] = \{x \in L \; : \; a \leq x \leq b\},$$

$$(a,b] = \{x \in L \; : \; a < x \leq b\},$$

$$[a,b) = \{x \in L \; : \; a \leq x < b\},$$

$$(a,b) = \{x \in L \; : \; a < x < b\}.$$

If a and b are incomparable, we use the notation $a \parallel b$.

2.2 Uninorms on bounded lattices

Here we give the definition and basic properties of uninorms on a lattice and some examples of constructions of uninorms on the lattice mentioned in the introduction of this chapter.

Definition 2.4. Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $T : L^2 \to L$ ($S : L^2 \to L$) is called a triangular norm (conorm) on L if it is increasing, commutative, associative and has the neutral element $e = 1$ ($e = 0$), i.e., $T(x, 1) = x$ ($S(x, 0) = x$) for all $x \in L$.

Definition 2.5 ([218]). Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $T : L^2 \to L$ ($S : L^2 \to L$) is called a t-subnorm (t-superconorm) on L if it is increasing, commutative, associative and $T(x, y) \leq x \wedge y$ ($S(x, y) \geq x \vee y$) for all $x, y \in L$.

Definition 2.6 (cf. [137]). Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L$. An operations $U : L^2 \to L$ is called a uninorm on L if it is increasing, commutative, associative and has the neutral element e.

Theorem 2.1. *Let $e \in L \setminus \{0, 1\}$ and U be a uninorm on L with the neutral element e. Then there exist a t-norm T on the lattice $([0, e], \leq_L)$ and a t-conorm S on the lattice $([e, 1], \leq_L)$ such that*

$$U(x,y) = \begin{cases} T(x,y) & \textit{if } x, y \in [0,e], \\ S(x,y) & \textit{if } x, y \in [e,1]. \end{cases} \tag{2.1}$$

Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L$. Denote $A(e) = [0, e) \times (e, 1] \cup (e, 1] \times [0, e)$ and $I_e = \{x \in L \mid x \parallel e\}$.

Proposition 2.1 (cf. [137]). *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$ and U a uninorm on L with the neutral element e. The following properties hold:*

i) $x \wedge y \leq U(x, y) \leq x \vee y$ *for* $(x, y) \in A(e)$,
ii) $U(x, y) \leq x$ *for* $(x, y) \in L \times [0, e]$,

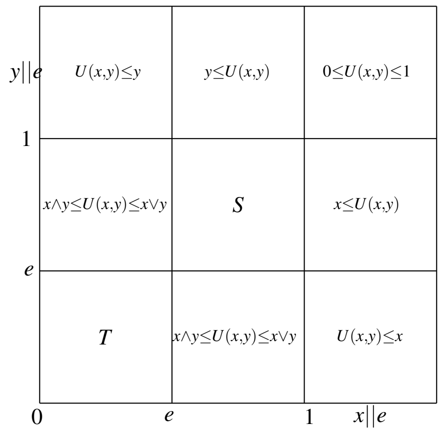

Fig. 2.2 The structure of uninorms on the lattice

iii) $U(x,y) \leq y$ *for* $(x,y) \in [0,e] \times L$,
iv) $x \leq U(x,y)$ *for* $(x,y) \in L \times [e,1]$,
v) $y \leq U(x,y)$ *for* $(x,y) \in [e,1] \times L$.

Below we present some construction of a uninorm on a bounded lattice.

Theorem 2.2 ([137]). *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $e \in L \setminus \{0,1\}$. *If* T_e *is a t-norm on* $[0,e]$ *and* S_e *is a t-conorm on* $[e,1]$ *then the operation* $U_t : L^2 \to L$ *defined as follows*

$$U_t(x,y) = \begin{cases} T_e(x,y) & if\ x,y \in [0,e], \\ x \vee y & if\ (x,y) \in [0,e] \times (e,1] \cup (e,1] \times [0,e], \\ y & if\ x \in [0,e],\ y \| e, \\ x & if\ y \in [0,e],\ x \| e, \\ 1 & otherwise \end{cases}$$

and $U_s : L^2 \to L$ *defined as follows*

$$U_s(x,y) = \begin{cases} S_e(x,y) & if\ x,y \in [e,1], \\ x \wedge y & if\ (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e), \\ y & if\ x \in [e,1],\ y \| e, \\ x & if\ y \in [e,1],\ x \| e, \\ 0 & otherwise \end{cases}$$

are uninorms on L.

Theorem 2.3 ([46]). *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $e \in L \setminus \{0,1\}$. *If* T_e *is a t-norm on* $[0,e]$ *and* S_e *is a t-conorm on* $[e,1]$ *then the operation* $U_t : L^2 \to L$ *defined as follows*

$$U_t(x,y) = \begin{cases} T_e(x,y) & if\ x,y \in [0,e], \\ y & if\ x \in [0,e],\ y\|e, \\ x & if\ y \in [0,e],\ x\|e, \\ x \vee y & otherwise \end{cases}$$

and $U_s : L^2 \to L$ *defined as follows*

$$U_s(x,y) = \begin{cases} S_e(x,y) & if\ x,y \in [e,1], \\ y & if\ x \in [e,1],\ y\|e, \\ x & if\ y \in [e,1],\ x\|e, \\ x \wedge y & otherwise \end{cases}$$

are uninorms on L.

Theorem 2.4 ([30, 54]). *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $e \in L \setminus \{0,1\}$. *If T is a t-norm on L and S is a t-conorm on L then the operation* $U_d : L^2 \to L$ *defined as follows*

$$U_d(x,y) = \begin{cases} T(x,y) & if\ x,y \in [0,e), \\ y & if\ x \in [0,e],\ y\|e, \\ x & if\ y \in [0,e],\ x\|e, \\ S(x,y) & if\ x,y \in (e,1], \\ S(x \vee e, y \vee e) & otherwise \end{cases}$$

and $U_c : L^2 \to L$ *defined as follows*

$$U_c(x,y) = \begin{cases} S(x,y) & if\ x,y \in (e,1], \\ y & if\ x \in [e,1],\ y\|e, \\ x & if\ y \in [e,1],\ x\|e, \\ T(x,y) & if\ x,y \in [0,e), \\ T(x \wedge e, y \wedge e) & otherwise \end{cases}$$

are uninorms on L.

Theorem 2.5 ([40]). *Let* $(L,\leq,0,1)$ *be a bounded lattice and* $e \in L \setminus \{0,1\}$, $a\|b$ *for all* $a \in I_e$, $b \in (0,e]$. *If* T_e *is a t-norm on* $[0,e]$ *such that* $T_e(a,b) > 0$ *for all* $a,b > 0$ *and* S_e *is a t-conorm on* $[e,1]$ *such that* $S_e(a,b) < 1$ *for all* $a,b < 1$ *then the operation* $U_t : L^2 \to L$ *defined as follows*

$$U_t(x,y)=\begin{cases}T_e(x,y) & if\ x,y\in[0,e],\\ S_e(x,y) & if\ x,y\in[e,1],\\ x & if\ y\in(0,e]\cup(e,1),\ x\|e,\\ y & if\ x\in(0,e]\cup(e,1),\ y\|e,\\ x\wedge y & if\ x,y\|e\ or\ x\|e, y=0\ or\ x=0, y\|e\\ & \ or\ x\in(e,1), y=0\ or\ x=0, y\in(e,1),\\ x\vee y & otherwise\end{cases}$$

is a uninorm on L.

2.3 Uninorms from the classes $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$

The classes $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ are the special classes of uninorms on the unit interval. Here we will present the generalization of these classes of uninorms on the lattice and their structure.

Definition 2.7 ([269]). Let $(L,\leq,0,1)$ be a bounded lattice and $e\in L\setminus\{0,1\}$. By $\mathscr{U}_{\min}$ we denote the class of all uninorms U on L with the neutral element e satisfying the following condition:

$$U(x,y)=y \text{ for all } (x,y)\in(e,1]\times(L\setminus[e,1]).$$

By $\mathscr{U}_{\max}$ we denote the class of all uninorms U on L with the neutral element e satisfying the following condition:

$$U(x,y)=y \text{ for all } (x,y)\in[0,e)\times(L\setminus[0,e]).$$

Remark 2.1. Using the notion

$$U(x,y)=x\wedge y \text{ for all } (x,y)\in(e,1]\times(L\setminus[e,1])$$

we obtain inequalities $y=U(e,y)\leq U(x,y)=x\wedge y\leq x$ for all $x\in(e,1]$ and $y\in I_e$. So, $y<x$ for all $x\in(e,1]$, $y\in I_e$, which will not allow to define the considered classes on any bounded lattice.

Theorem 2.6 ([269]). *Let $(L,\leq,0,1)$ be a bounded lattice, $e\in L\setminus\{0,1\}$ and U be a binary operation on L. Then $U\in\mathscr{U}_{\min}$ if and only if there exist a t-conorm S on $[e,1]$ and t-subnorm T on $L\setminus[e,1]$ such that*

$$U(x,y)=\begin{cases}S(x,y) & if\ x,y\in[e,1],\\ y & if\ x\in[e,1],\ y\in L\setminus[e,1],\\ x & if\ x\in L\setminus[e,1],\ y\in[e,1],\\ T(x,y) & otherwise.\end{cases}$$

Theorem 2.7 ([269]). *Let $(L,\leq,0,1)$ be a bounded lattice and $e\in L\setminus\{0,1\}$, U be a binary operation on L. Then $U\in\mathscr{U}_{\max}$ if and only if there exist a t-norm T on $[0,e]$ and t-superconorm S on $L\setminus[0,e]$ such that*

$$U(x,y)=\begin{cases}T(x,y) & if\ x,y\in[0,e],\\ y & if\ x\in[0,e],\ y\in L\setminus[0,e],\\ x & if\ x\in L\setminus[0,e],\ y\in[0,e],\\ S(x,y) & otherwise.\end{cases}$$

A detailed description of these classes and a discussion of other generalizations of the $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ classes can be found in paper [269].

2.4 Idempotent uninorms

Definition 2.8 ([46]). Let $(L,\leq,0,1)$ be a bounded lattice, $e\in L\backslash\{0,1\}$ and U a uninorm on L with the neutral element e. U is called an idempotent uninorm if $U(x,x)=x$ for all $x\in L$.

Proposition 2.2 ([46]). *Let $(L,\leq,0,1)$ be a bounded lattice, $e\in L\backslash\{0,1\}$ and U an idempotent uninorm on L with the neutral element e. Then it holds:*

i) $U(x,y)=x\wedge y$ for all $(x,y)\in[0,e]^2$,
ii) $U(x,y)=x\vee y$ for all $(x,y)\in[e,1]^2$.

Proposition 2.3 ([43]). *Let $(L,\leq,0,1)$ be a bounded lattice, $e\in L\backslash\{0,1\}$ and U an idempotent uninorm on L with the neutral element e. Then it holds:*

i) $U(x,y)\in\{x,y\}$ or $U(x,y)\in I_e$ for all $(x,y)\in A(e)$,
ii) $U(x,y)=x\wedge y$ or $U(x,y)\in I_e$ for all $x\in[0,e]$ and $y\in I_e$,
iii) $U(x,y)=x\vee y$ or $U(x,y)\in I_e$ for all $x\in[e,1]$ and $y\in I_e$,
iv) $U(x,y)=x\wedge y$ or $U(x,y)\in I_e$ for all $y\in[0,e]$ and $x\in I_e$,
v) $U(x,y)=x\vee y$ or $U(x,y)\in I_e$ for all $y\in[e,1]$ and $x\in I_e$,
vi) $U(x,y)\in\{x\wedge y,x\vee y\}$ or $U(x,y)\in I_e$ for all $x\in I_e$ and $y\in I_e$.

One way to construct idempotent uninorms was presented by Çaylı.

Theorem 2.8 ([46]). *Let $(L,\leq,0,1)$ be a bounded lattice and $e\in L\setminus\{0,1\}$. Then the operation $U_{t_\wedge}:L^2\to L$ defined as follows*

$$U_{t_\wedge}(x,y)=\begin{cases}x\wedge y & if\ x,y\in[0,e],\\ y & if\ x\in[0,e],\ y\|e,\\ x & if\ y\in[0,e],\ x\|e,\\ x\vee y & otherwise\end{cases}$$

and $U_{s_\vee}:L^2\to L$ defined as follows

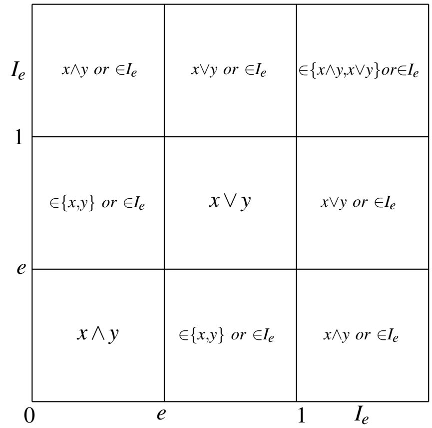

Fig. 2.3 Structure of idempotent uninorms

$$U_{s_\vee}(x,y)=\begin{cases} x\vee y & if\ x,y\in[e,1],\\ y & if\ x\in[e,1],\ y\|e,\\ x & if\ y\in[e,1],\ x\|e,\\ x\wedge y & otherwise\end{cases}$$

are the greatest and the least idempotent uninorm on L with neutral element e.

Remark 2.2. The constructions from Theorem 2.2 give an idempotent uninorm if and only if all elements of the lattice L are comparable to the neutral element and T_e, S_e are idempotent. This is a slight difference from the statement in the paper [46] where the authors write that this construction never produces an idempotent uninorm.

Proposition 2.4 ([43]). *Let $(L,\leq,0,1)$ be a bounded lattice, U an idempotent uninorm on L with the neutral element $e\in L\backslash\{0,1\}$ such that all $x\in L$ are comparable with e. Then, $U(x,y)\in\{x\wedge y, x\vee y\}$ for all $(x,y)\in L^2$.*

Directly from Proposition 2.2, we see that an idempotent uninorm can not be locally internal in the sense of Definition 1.17. In addition, the following example shows that it is not possible to modify the term locally internal according to the above proposition.

Example 2.2 ([43]). Given a bounded lattice $L=\{0,x,y,e,z,t,1\}$ with the order in Figure 2.4, define a mapping $U: L^2\to L$ by Table 2.1. Then U is an idempotent uninorm on L with the neutral element e and $U(z,x)=t$.

Proposition 2.5 ([43]). *Let $(L,\leq,0,1)$ be a lattice, U an idempotent uninorm on L with the neutral element $e\in L\backslash\{0,1\}$ such that all $x\in L$ are comparable with e. For $x,y,z\in L$ such that $x,y\geq e$, $x\parallel y$ and $z\leq e$, it may be possible only one of the following conditions:*

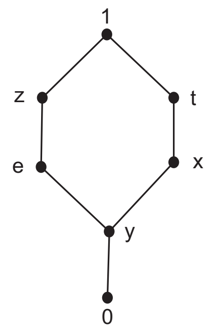

Fig. 2.4 The lattice given in Example 2.2

U	0	y	x	e	z	t	1
0	0	0	0	0	0	0	0
y	0	y	y	y	y	t	t
x	0	y	x	x	t	t	t
e	0	y	x	e	z	t	1
z	0	y	t	z	z	t	1
t	0	t	t	t	t	t	t
1	0	t	t	1	1	t	1

Table 2.1 The idempotent uninorm given in Example 2.2

i) If $U(x,z) = z$, then $U(y,z) = z$, $U(x \vee y,z) = z$ and $U(x \wedge y,z) = z$.
ii) If $U(x,z) = x$, then $U(y,z) = y$ and $U(x \vee y,z) = x \vee y$.

Proposition 2.6 ([43]). *Let $(L,\leq,0,1)$ be a lattice, U an idempotent uninorm on L with the neutral element $e \in L \backslash \{0,1\}$ such that all $x \in L$ are comparable with e. If $U(x,z) = z$ for $x,y \geq e$, $x \parallel y$ and $z \leq e$, then $U(a,z) = U(b,z) = U(a \wedge b,z) = U(a \vee b,z) = z$ for arbitrary $a,b \geq e$, $a \parallel b$ such that $x \wedge y = a \wedge b$.*

Proposition 2.7 ([43]). *Let $(L,\leq,0,1)$ be a bounded lattice, U an idempotent uninorm on L with the neutral element $e \in L \backslash \{0,1\}$ such that all $x \in L$ are comparable with e. If $U(x,z) = x$ for $x,y \geq e$, $x \parallel y$ and $z \leq e$, then $U(a,z) = a$, $U(b,z) = b$, $U(a \wedge b,z) = a \wedge b$ and $U(a \vee b,z) = a \vee b$ for arbitrary $a,b \geq e$, $a \parallel b$ such that $x \vee y = a \wedge b$.*

Proposition 2.8 ([43]). *Let $(L,\leq,0,1)$ be a bounded lattice, U an idempotent uninorm on L with the neutral element $e \in L \backslash \{0,1\}$ such that all $x \in L$ are comparable with e. If $U(x,z) = z$ for $x,y \geq e$, $x \parallel y$ and $z \leq e$, then $U(a,z) = U(b,z) = U(a \wedge b,z) = U(a \vee b,z) = z$ for arbitrary $a,b \geq e$, $a \parallel b$ such that $x \wedge y = a \vee b$.*

In a similar way as Proposition 2.6 we can obtain the following property.

Proposition 2.9 ([43]). *Let $(L,\leq,0,1)$ be a lattice, U an idempotent uninorm on L with the neutral element $e \in L \backslash \{0,1\}$ such that all $x \in L$ are comparable with e. If $U(x,z) = x$ for $x,y \geq e$, $x \parallel y$ and $z \leq e$, then $U(a,z) = a$, $U(b,z) = b$ and $U(a \vee b,z) = a \vee b$ for arbitrary $a,b \geq e$, $a \parallel b$ such that $x \wedge y = a \wedge b$.*

Proposition 2.10 ([43]). *Let $(L,\leq,0,1)$ be a lattice, U an idempotent uninorm on L with the neutral element $e \in L\backslash\{0,1\}$ such that all $x \in L$ are comparable with e. If $U(x,z) = z$ for $x,y \geq e$, $x \parallel y$ and $z \leq e$, then $U(a,z) = z$, $U(b,z) = z$, $U(a \wedge b,z) = z$ and $U(a \vee b,z) = z$ for arbitrary $a,b \geq e$, $a \parallel b$ such that $x \vee y = a \vee b$.*

Proposition 2.11 ([43]). *Let $(L,\leq,0,1)$ be a lattice, U an idempotent uninorm on L with the neutral element $e \in L\backslash\{0,1\}$ such that all $x \in L$ are comparable with e. If $U(x,z) = x$ for $x,y \geq e$, $x \parallel y$ and $z \leq e$, then $U(a,z) = a$, $U(b,z) = b$ and $U(a \vee b,z) = a \vee b$ for arbitrary $a,b \geq e$, $a \parallel b$ such that $x \vee y = a \vee b$.*

We can also obtain dual results to Propositions 2.5–2.11(see [43]), and consequently we obtain the next result.

Theorem 2.9 ([43]). *Let $(L,\leq,0,1)$ be a bounded lattice, U an idempotent uninorm on L with the neutral element $e \in L\backslash\{0,1\}$ such that all $x \in L$ comparable with e. Then the set*

$$A_x = \{y \in L : y||x \text{ or } \exists_z\,(z||x \text{ and } z||y\,)\}\,.$$

is a subset of $[0,e]$ or $[e,1]$. Moreover
i) If $U(y,z) = y$ for $y \in A_x$, where $x \leq e \leq z$ or $z \leq e \leq x$, then $U(a,z) = a$ for arbitrary $a \in A_x$.
ii) If $U(y,z) = z$ for $y \in A_x$, where $x \leq e \leq z$ or $z \leq e \leq x$, then $U(a,z) = z$ for arbitrary $a \in A_x$.

2.4.1 Separating function of idempotent uninorms on some types of lattices

In this section we will consider the structure of uninorms under some assumptions, that L is a finite lattice and all elements are comparable with e. In the beginning we will present the characterization of idempotent uninorms, assuming the linearity of the lattice L. Then, using previous results, we present a characterization of idempotent uninorms without the assumption of linearity. The presented results are a supplemented and corrected version of the paper [44].

2.4.1.1 The case when L is a chain

In this part we will adopt the results from the paper [62]. First we define the separating function and mention some of its properties.

Lemma 2.1. *Let $(L,\leq,0,1)$ be a chain, U be an idempotent uninorm on L with the neutral element $e \in L\backslash\{0,1\}$ and $Card\,(L) < \aleph_0$. Then the function $g : L \to L$ defined as follows*

$$g(x) = \begin{cases} 0 & \textit{if } x > e \textit{ and} \\ & \min\{z \in L:\ U(x,z) = \max(x,z)\} = 0 \\ \max\{z \in L : U(x,z) = \min(x,z)\} & \textit{otherwise} \end{cases} \tag{2.2}$$

has the following properties:

i) $g(e) = e$.
ii) For all $x < e$, *it holds that* $g(x) \geq e$ *and* $U(x,y) = x \wedge y$ *whenever* $y \leq g(x)$.
iii) For all $x > e$, *it holds that* $g(x) \leq e$ *and* $U(x,y) = x \vee y$ *whenever* $y > g(x)$.

Proof. By easy verification we see that the function $g : L \to L$ defined by (2.2) is well defined. Moreover to prove i) we have

$$\begin{aligned} g(e) &= \max\{z \in L:\ U(e,z) = \min(e,z)\} \\ &= \max\{z \in L:\ z = \min(e,z)\} \\ &= \max\{z \in L:\ z \leq e\} = e. \end{aligned}$$

ii) For all $x < e$, we have $U(x,e) = x = \min(x,e)$ and

$$g(x) = \max\{z \in L:\ U(x,z) = \min(x,z)\} \geq e.$$

Moreover, because of Proposition 2.2 (i) it is enough to consider the condition $x < e \leq y \leq g(x)$. By monotonicity of U and definition of g, we obtain $U(x,y) \leq U(x,g(x)) = x$ and $U(x,y) \geq U(x,e) = x$. So, $U(x,y) = x = x \wedge y$.
iii) It can be proved similarly as (ii). □

Theorem 2.10. *Let* $(L, \leq, 0, 1)$ *be a chain,* U *be an idempotent uninorm on* L *with the neutral element* $e \in L \backslash \{0,1\}$ *and* $Card\,(L) < \aleph_0$. *Then there exists a decreasing function* $g : L \to L$ *with* $g(e) = e$ *such that*

$$U\,(x,y) = \begin{cases} x \wedge y & \textit{if } y \leq g(x) \textit{ and } g(x) > 0, \\ x \vee y & \textit{if } y > g(x), \\ x \wedge y \textit{ or } x \vee y & \textit{otherwise.} \end{cases} \tag{2.3}$$

Proof. Consider the function $g : L \to L$ defined by (2.2). By Lemma 2.1, we obtain (2.3). The monotonicity of U immediately implies that g is decreasing. □

In [174], it is shown that any idempotent, associative, increasing binary operation U on the unit interval with the neutral element e must be commutative, except perhaps in points $(x, g(x))$ such that $g^2(x) = x$ (see Theorem 1.27). This condition is not true in the considered case as it is illustrated by Example 2.3.

Example 2.3. Let $L = \{0, a, b, e, c, d, 1\}$ be a chain such that $0 \leq a \leq b \leq e \leq c \leq d \leq 1$ and consider the function $g : L \to L$ defined as

$$g(x) = \begin{cases} d & \text{if } x = 0, \\ c & \text{if } x = a \text{ or } x = b, \\ e & \text{if } x = e, \\ a & \text{if } x = c \text{ or } x = d, \\ 0 & \text{if } x = 1. \end{cases}$$

Furthermore, let U be the binary operation on L defined as

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y \leq g(x), \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Obviously, U is idempotent, associative, increasing and has the neutral element e. However, U is not commutative, as it holds for instance that $U(a,d) = d \neq a = U(d,a)$, farther $g(d) = a$ and $g(g(d)) = c \neq d$.

Definition 2.9. Let $g : L \to L$ be a decreasing function. Its completed graph F_g is defined as the subset of L^2 in the following way:

$$\begin{aligned} F_g = & (\{0\} \times [g(0), 1]) \cup (\{1\} \times [0, g(1)]) \\ & \cup \{(x,y) \in [0,t] \times [0,1] : \ g(k) \leq y \leq g(x), \\ & t = \max\{s \in L : \ s < 1\},\ k = \min\{l \in L : \ l > x\}\}. \end{aligned}$$

Definition 2.10. A subset F of L^2 is said to be symmetrical if for all $(x,y) \in L^2$ it holds that

$$(x,y) \in F \Leftrightarrow (y,x) \in F.$$

Definition 2.11. A decreasing function $g : L \to L$ is said to be symmetrical if its completed graph F_g is symmetrical.

Example 2.4. Let $L = \{0,a,e,b,1\}$, where $0 \leq a \leq e \leq b \leq 1$ and consider the function $g : L \to L$ defined as

$$g(x) = \begin{cases} 1 & \text{if } x = 0, \\ e & \text{if } x \in \{a,e\}, \\ 0 & \text{if } x \in \{b,1\}. \end{cases}$$

The graph of g is the set

$$\{(0,1), (a,e), (e,e), (b,0), (1,0)\}$$

whereas its completed graph F_g is given by

$$F_g = \{(0,1), (1,0), (0,e), (0,b), (a,e), (e,0), (e,a), (e,e), (b,0)\}.$$

It is clear that F_g is symmetrical, and hence also g, whereas the graph of g is not symmetrical.

Lemma 2.2. *Let $g : L \to L$ be a decreasing function. The following statements are equivalent:*

i) g is symmetrical.
ii) $g(x) = 0$ for all $x > g(0)$ and the set $F_{g,0}$ is symmetrical, where

$$F_{g,0} = \left\{ (x,y) \in [0,g(0)]^2 : \ g(k) \leq y \leq g(x), \ k = \min\{l \in L : \ l > x\} \right\}.$$

iii) $g(x) = 1$ for all $x < g(1)$ and the set $F_{g,1}$ is symmetrical, where

$$F_{g,1} = \left\{ (x,y) \in [g(1),1]^2 : \ g(k) \leq y \leq g(x), \ k = \min\{l \in L : \ l > x\} \right\}.$$

Lemma 2.3. *Let $g : L \to L$ be a decreasing function such that $g(0) < 1$. The following statements are equivalent:*

i) g is symmetrical.
ii) $g(x) = 0$ for all $x > g(0)$ and $g(g(x)) \geq x$ for all $x \in [0,g(0)]$.
iii) $g(x) = 0$ for all $x > g(0)$ and for all $(x,y) \in [0,g(0)]^2$ it holds that

$$y \leq g(x) \Leftrightarrow x \leq g(y).$$

Lemma 2.4. *Let $e \in L$ such that $0 < e < 1$ and let $g^* : [0,e] \to [e,1]$ be a decreasing function such that $g^*(e) = e$. Then there exists exactly one symmetrical extension of g^*, $\bar{g} : L \to L$, given by*

$$\bar{g}(x) = \begin{cases} g^*(x) & x \leq e, \\ \max\{z \in [0,e] : \ g^*(z) \geq x\} & e < x \leq g^*(0), \\ 0 & x > g^*(0). \end{cases} \tag{2.4}$$

Remark 2.3. Similarly as in the paper [174], any commutative, increasing binary operation on L that is locally internal (i.e., such that $U(x,y) \in \{x,y\}$ for all $(x,y) \in L^2$) must be associative.

Theorem 2.11. *A binary operation U on L with the neutral element $0 < e < 1$ is an idempotent uninorm if and only if there exists a decreasing function $g^* : [0,e] \to [e,1]$ with $g^*(e) = e$ such that*

$$U(x,y) = \begin{cases} \min(x,y) & if\ y \leq \bar{g}(x)\ and\ x \leq \bar{g}(0), \\ \max(x,y) & otherwise \end{cases} \tag{2.5}$$

where $\bar{g}$ is the unique symmetrical extension of g^* given by (2.4).

Example 2.5. Consider the chain $L = \{0,a,e,b,1\}$ such that $0 \leq a \leq e \leq b \leq 1$ and the function $g^* : [0,e] \to [e,1]$ defined as

$$g^*(x) = \begin{cases} 1 & \text{if } x < e, \\ e & \text{if } x = e \end{cases}$$

and its symmetrical extension $\bar{g} : L \to L$ given by

$$\bar{g}(x) = \begin{cases} 1 & \text{if } x < e, \\ e & \text{if } x = e, \\ a & \text{otherwise.} \end{cases} \tag{2.6}$$

The uninorm given by (2.5) with the function $\bar{g}$ given by (2.6) is an idempotent uninorm in the class $\mathscr{U}_{\min}$.

Similarly, consider the constant function $g^* : [0,e] \to [e,1]$ defined by $g^*(x) = e$, then its symmetrical extension $\bar{g} : L \to L$ is given by

$$\bar{g}(x) = \begin{cases} e & \text{if } x \le e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

The uninorm given by (2.5) with the function $\bar{g}$ given by (2.7) is an idempotent uninorm in the class $\mathscr{U}_{\max}$.

Remark 2.4. If the function g will be defined on the edge of the values of the function max then in a similar way as above we can obtain the description of the idempotent uninorm.

Lemma 2.5. *Let $(L, \le, 0, 1)$ be a chain, U be an idempotent uninorm on L with the neutral element $e \in L \backslash \{0,1\}$ and $Card(L) < \aleph_0$. Then the function $g : L \to L$ defined by*

$$g^{\vee}(x) = \begin{cases} 1 & \textit{if } x < e \textit{ and } \\ & \max\{z \in L : \ U(x,z) = \min(x,z)\} = 1, \\ \min\{z \in L : \ U(x,z) = \max(x,z)\} & \textit{otherwise} \end{cases} \tag{2.8}$$

has the following properties:
i) $g^{\vee}(e) = e$.
ii) For all $x < e$, it holds that $g^{\vee}(x) \ge e$ and $U(x,y) = x \wedge y$ whenever $y < g^{\vee}(x)$.
iii) For all $x > e$, it holds that $g^{\vee}(x) \le e$ and $U(x,y) = x \vee y$ whenever $y \ge g^{\vee}(x)$.

Theorem 2.12. *Let $(L, \le, 0, 1)$ be a chain, U be an idempotent uninorm on L with the neutral element $e \in L \backslash \{0,1\}$ and $Card(L) < \aleph_0$. Then there exists a decreasing function $g^{\vee} : L \to L$ with $g^{\vee}(e) = e$ such that*

$$U(x,y) = \begin{cases} x \wedge y & \textit{if } y < g^{\vee}(x), \\ x \vee y & \textit{if } y \ge g^{\vee}(x) \textit{ and } g^{\vee}(x) < 1, \\ x \wedge y \textit{ or } x \vee y & \textit{otherwise.} \end{cases} \tag{2.9}$$

Lemma 2.6. *Let $e \in L$ such that $0 < e < 1$ and $g^{\vee} : [e,1] \to [0,e]$ be a decreasing function such that $g^{\vee}(e) = e$. Then there exists exactly one symmetrical extension of $g^{\vee}$, $\bar{g}^{\vee} : L \to L$, given by*

$$\bar{g}^{\vee}(x) = \begin{cases} g^{\vee}(x) & x \geq e, \\ \max\{z \in [e,1] : \; g^{\vee}(z) \geq x\} & g^{\vee}(1) \leq x < e, \\ 1 & x < g^{\vee}(1). \end{cases} \quad (2.10)$$

Theorem 2.13. *A binary operation U on L with the neutral element $0 < e < 1$ is an idempotent uninorm if and only if there exists a decreasing function $g^{\vee} : [e,1] \to [0,e]$ with $g^{\vee}(e) = e$ such that*

$$U(x,y) = \begin{cases} \max(x,y) & if \; y \geq \bar{g}^{\vee}(x) \; and \; x \geq \bar{g}^{\vee}(1), \\ \min(x,y) & otherwise, \end{cases} \quad (2.11)$$

where $\bar{g}^{\vee}$ is the unique symmetrical extension of $g^{\vee}$ given by (2.10).

Remark 2.5. Note the relationship between the functions $\bar{g}$ and $\bar{g}^{\vee}$. For all $x \in [g(1),e) \cup (e,g(0)]$ the value $\bar{g}^{\vee}(x)$ is direct successor of $\bar{g}(x)$. On the rest of the lattice both functions are equal.

2.4.1.2 The case when L is a nonlinearly ordered lattice

In this part we consider more general lattice – without assumption about linearity. That is, we will assume that $(L, \leq, 0, 1)$ is a bounded lattice, with such an element $e \in L \backslash \{0,1\}$ that all elements of the lattice are comparable with e and in addition $Card(L) < \aleph_0$.

Consider the following example.

Example 2.6. Given a lattice $L = \{0,a,e,b,c,d,1\}$ with the order given in Figure 2.5, define a mapping $U : L^2 \to L$ by Table 2.2. Let us consider the function g given by (2.2). Then the set $B_a = \{z \in L : \; U(x,z) = \min(a,z)\}$ does not have the greatest element. This means that the function g is not well defined.

But, this set has the maximal elements c and d. If we allow to replace the greatest element by the maximal element, we will not have uniqueness. In addition, the operation U given by (2.3) will not be specified for points (x,y), where y is another maximal element of B_x different than $g(x)$.

So we need to define another function which separates values min and max. We will use functions given by (2.2) and (2.8), and the relationship between them.

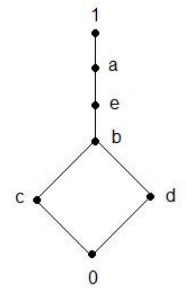

Fig. 2.5 The lattice given in Example 2.6

U	0	d	c	b	e	a	1
0	0	0	0	0	0	0	0
d	0	d	0	d	d	d	d
c	0	0	c	c	c	c	c
b	0	d	c	b	b	a	1
e	0	d	c	b	e	a	1
a	0	d	c	a	a	a	1
1	0	d	c	1	1	1	1

Table 2.2 The idempotent uninorm given in Example 2.6

Remark 2.6. Let $(L,\leq,0,1)$ be a bounded lattice, U be an idempotent uninorm on L with the neutral element $e\in L\backslash\{0,1\}$ such that all $x\in L$ are comparable with e and $Card\,(L)<\aleph_0$. Then the function $g:L\to L$ defined by

$$g\,(x)=\begin{cases}\max\{z\in L:\ U(x,z)=\min(x,z)\} & if\ x\leq e,\\ \min\{z\in L:\ U(x,z)=\max(x,z)\} & otherwise\end{cases} \tag{2.12}$$

separate the values min and max.

Lemma 2.7. *Let $(L,\leq,0,1)$ be a bounded lattice, U be an idempotent uninorm on L with the neutral element $e\in L\backslash\{0,1\}$ such that all $x\in L$ are comparable with e and $Card\,(L)<\aleph_0$. Then the function $g:L\to L$ defined by* (2.12) *has the following properties:*
i) $g(e)=e$.
ii) For all $x<e$, it holds that $g(x)\geq e$ and $U(x,y)=x\wedge y$ whenever $y\leq g(x)$.
iii) For all $x>e$, it holds that $g(x)\leq e$ and $U(x,y)=x\vee y$ whenever $y\geq g(x)$.

Proof. It is similar as the proofs of Lemma 2.1 and Lemma 2.5. □

Lemma 2.8. *Let $(L,\leq,0,1)$ be a lattice, U be an idempotent uninorm on L with the neutral element $e\in L\backslash\{0,1\}$ such that all $x\in L$ are comparable with e and $Card\,(L)<\aleph_0$. If we consider $g:L\to L$ defined by* (2.12)*, then*
i) It does not exist $y\in L$ such that $y\parallel g(x)$ for all $x<e$.
ii) It does not exist $y\in L$ such that $y\parallel g(x)$ for all $x>e$.

Proof. i) Suppose that there exists $y \in L$ such that $y \parallel g(x)$ for some $x < e$. By Proposition 2.4, $U(x,y) \in \{x,y\}$. If $U(x,y) = x$, then by Proposition 2.5, $U(x, y \vee g(x)) = x$. So, we obtain that

$$g(x) = \max\{z \in L : \ U(x,z) = \min(x,z)\} \geq y \vee g(x) > g(x).$$

This is a contradiction. If $U(x,y) = y$, by Proposition 2.5, $U(x,g(x)) = g(x)$. This is contradiction with Lemma 2.1 (ii).
ii) It can be proved similarly as (i). □

Theorem 2.14. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice, U be an idempotent uninorm on L with the neutral element* $e \in L \backslash \{0,1\}$ *such that all* $x \in L$ *are comparable with e and Card* $(L) < \aleph_0$*. Then there exists a decreasing function* $g : L \to L$ *with* $g(e) = e$ *such that*

$$U(x,y) = \begin{cases} x \wedge y & if \ (y \leq g(x) \ and \ x \leq e) \ or \ (y < g(x) \ and \ x > e), \\ x \vee y & if \ (y > g(x) \ and \ x \leq e) \ or \ (y \geq g(x) \ and \ x > e). \end{cases} \tag{2.13}$$

Proof. Consider the function $g : L \to L$ defined by (2.12). By using Proposition 2.4 and Lemma 2.7, we obtain that U is given by (2.13). The monotonicity of U immediately implies that g is decreasing. □

Example 2.7. Let $L = \{0,a,b,c,d,r,q,e,s,z,t,l,h,m,p,1\}$ with the order given by Figure 2.6 be the bounded lattice. For an idempotent uninorm U on L defined by Table 2.3 the separating function $g : L \to L$ is defined as

$$g(x) = \begin{cases} 0 & x = 1, \\ a & x = p, \\ b & x \in \{t,l,h,m\}, \\ q & x = \{s,z\}, \\ e & x \in \{e,q\}, \\ s & x \in \{b,c,d,r\}, \\ m & x = a, \\ 1 & x = 0. \end{cases} \tag{2.14}$$

Unfortunately, the function g has no such properties as the function g given by (2.2). In the next part of this section we present the connection between both mentioned functions. First observe, that both functions have the same definition on $[0,e]$. Moreover, directly by Lemma 2.8 we see, that for the subset K of L such that $e \in K \backslash \{0,1\}$ and each element of K is comparable with all elements of L (that is K is a chain) we can take the decreasing function $g^* : K \to K$ defined by (2.2) or $g^{\vee} : K \to K$ defined by (2.8) and we obtain the characterization of U on the lattice K.

Remark 2.7. Let us consider the set

$$K = \{x \in L : \ x \text{ is comparable with all } y \in L\}. \tag{2.15}$$

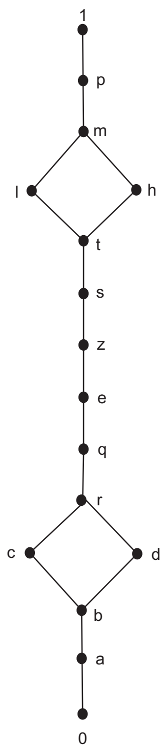

Fig. 2.6 The lattice given in Example 2.7

U	0	a	b	c	d	r	q	e	z	s	t	l	h	m	p	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a	a	a	a	a	a	a	p	1
b	0	a	b	b	b	b	b	b	b	b	t	l	h	m	p	1
c	0	a	b	c	b	c	c	c	c	c	t	l	h	m	p	1
d	0	a	b	b	d	d	d	d	d	d	t	l	h	m	p	1
r	0	a	b	c	d	r	r	r	r	r	t	l	h	m	p	1
q	0	a	b	c	d	r	q	q	z	s	t	l	h	m	p	1
e	0	a	b	c	d	r	q	e	z	s	t	l	h	m	p	1
z	0	a	b	c	d	r	z	z	z	s	t	l	h	m	p	1
s	0	a	b	c	d	r	s	s	s	s	t	l	h	m	p	1
t	0	a	t	t	t	t	t	t	t	t	t	l	h	m	p	1
l	0	a	l	l	l	l	l	l	l	l	l	l	m	m	p	1
h	0	a	h	h	h	h	h	h	h	h	h	m	h	m	p	1
m	0	a	m	m	m	m	m	m	m	m	m	m	m	m	p	1
p	0	p	p	p	p	p	p	p	p	p	p	p	p	p	p	1
1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 2.3 The idempotent uninorm given in Example 2.7

Lemma 2.9. *Let* $(L, \leq, 0, 1)$ *be a distributive lattice, U be an idempotent uninorm on L with the neutral element* $e \in L \backslash \{0, 1\}$*, all* $x \in L$ *be comparable with e,* $Card\,(L) < \aleph_0$ *and K given by* (2.15). *Then* $U^* = U|_K$ *is a uninorm on K.*

Question 2.1. Let $(L, \leq, 0, 1)$ be a bounded lattice, all $x \in L$ be comparable with e and $Card(L) < \aleph_0$, U be a uninorm on L. Then U is a uninorm on K with the decreasing and symmetrical function $g_K : K \to K$ such that $g_K(e) = e$. Can we always define the decreasing function $\widetilde{g}$ on L with $\widetilde{g}(e) = e$ such that $\widetilde{g}|K = g_K$ and operation U given by (2.5) or (2.13) using the function $\widetilde{g}$ is a uninorm?

Remark 2.8. Firstly, if we need symmetrical extension of the function g^*, then we should use the construction (2.5), but taking into account Example 2.6 we can not construct the symmetrical extension.

So, using the characterization of the uninorm on K we will be able to construct the function g which allows us to characterize the idempotent uninorm on L.

Theorem 2.15. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice, all* $x \in L$ *be comparable with* e *(*$0 < e < 1$*),* $Card(L) < \aleph_0$ *and* K *be defined by* (2.15). *If there exists a decreasing function* $g : K|_{[0,e]} \to K|_{[e,1]}$ *with* $g(e) = e$*, then the binary operation* U *given by*

$$U(x,y) = \begin{cases} x \wedge y & if \ \left(y \leq \widetilde{g}(x) \ and \ x \leq e\right) or \ \left(y < \widetilde{g}(x) \ and \ x > e\right), \\ x \vee y & if \ \left(y > \widetilde{g}(x) \ and \ x \leq e\right) or \ \left(y \geq \widetilde{g}(x) \ and \ x > e\right) \end{cases} \tag{2.16}$$

is an idempotent uninorm on L *with the neutral element* $e \in L \backslash \{0, 1\}$*, where* $\widetilde{g}$ *is an extension of* g*.*

Proof. First we present the construction of $\widetilde{g}$. Let g be a decreasing function $g : K|_{[0,e]} \to K|_{[e,1]}$ with $g(e) = e$. By Lemma 2.4 there exists exactly one symmetrical extension $\bar{g}$ of g. Let us define a decreasing function $g^{\vee} : K|_{[e,1]} \to K|_{[0,e]}$ as follows

$$g^{\vee}(x) = \begin{cases} e & if \ x = e, \\ \bar{g}(x) & if \ x > \bar{g}(0), \\ \min\left\{z \in [0,e] \cap K : \ \bar{g}(x) < z\right\} & otherwise. \end{cases} \tag{2.17}$$

Using Remark 2.5 we obtain some additional connection between functions g and $g^{\vee}$. Let now

$$\widetilde{g}(x) = \begin{cases} e & if \ x = e, \\ g(x) & if \ x \in [0,e] \cap K, \\ g^{\vee}(x) & if \ x \in [e,1] \cap K, \\ g(\max\{z \in [0,e] : \ z \in K \text{ and } x > z\}) & if \ x \in [0,e] \setminus K, \\ g^{\vee}(\min\{z \in [e,1] : \ z \in K \text{ and } x < z\}) & if \ x \in [e,1] \setminus K. \end{cases} \tag{2.18}$$

First observe that function $\widetilde{g}$ is decreasing. Next we will prove that U is a uninorm.
i) Monotonicity: We prove that if $x \leq y$ then for all $z \in L$, $U(x,z) \leq U(y,z)$. The proof is split into all possible cases.

1. Let $x \leq e$.

1.1. $y \leq e$,

1.1.1. $z \leq e$,

$$U(x,z) = x \wedge z \leq y \wedge z = U(y,z)$$

1.1.2. $z > e$, then $U(x,z) \in \{x,z\}$ and $U(y,z) \in \{y,z\}$. If $U(x,z) = x$, $U(x,z) \leq U(y,z)$. If $U(x,z) = z$, then $z > \widetilde{g}(x)$. Since $x \leq y$ and $\widetilde{g}$ is decreasing function, $\widetilde{g}(x) \geq \widetilde{g}(y)$. So, $z > \widetilde{g}(y)$. Hence, $U(y,z) = z$. In this case, we have $U(x,z) \leq U(y,z)$.

1.2. $y > e$,

1.2.1. $z \leq e$, then $U(x,z) = x \wedge z$ and $U(y,z) \in \{y,z\}$. In this case, we have $U(x,z) \leq U(y,z)$.

1.2.2. $z > e$, then $U(x,z) \in \{x,z\}$ and $U(y,z) = y \vee z$. In this case, we have $U(x,z) \leq U(y,z)$.

2. Let $x > e$. Then $y > e$.

2.1. $z \leq e$, then $U(x,z) \in \{x,z\}$ and $U(y,z) \in \{y,z\}$. If $U(x,z) = z$, $U(x,z) \leq U(y,z)$. If $U(x,z) = x$, then $z \geq \widetilde{g}(x)$. Since $x \leq y$ and $\widetilde{g}$ is decreasing function, $\widetilde{g}(x) \geq \widetilde{g}(y)$. So, $z \geq \widetilde{g}(y)$. Hence, $U(y,z) = y$. In this case, we have $U(x,z) \leq U(y,z)$.

2.2. $z > e$,

$$U(x,z) = x \vee z \leq y \vee z = U(y,z)$$

ii) Commutativity: We demonstrate that $U(x,y) = U(y,x)$ for all $x,y \in L$.

If $(x,y) \in [0,e]^2$, $U(x,y) = x \wedge y = y \wedge x = U(y,x)$. If $(x,y) \in [e,1]^2$, $U(x,y) = x \vee y = y \vee x = U(y,x)$. And if $(x,y) \in K^2$, by Theorem 2.11 and Remark 2.5 $U(x,y) = U(y,x)$. Thus, we only discuss the condition $(x,y) \in A(e)$ and at least one of its elements belong to the set K. For this, we define $b^x = \min\{t \in K \mid t > x\}$, $b_x = \max\{t \in K \mid t < x\}$ and we have $b_x, b^x \in K$.

Let $x \leq e$ and $y > e$.

1. $x \in K$ and $y \notin K$,

1.1. $y \leq \widetilde{g}(x)$, then $U(x,y) = x \wedge y = x$. Since $b^y \leq \widetilde{g}(x)$ and $b^y \in K$, using commutativity of U on K, $U(b^y,x) = U(x,b^y) = x \wedge b^y = x$. By the monotonicity of U, $U(y,x) \leq U(b^y,x) = x$. Since $U(y,x) \in \{y,x\}$, we have $U(y,x) = x$ and so, $U(x,y) = U(y,x)$.

1.2. $y > g(x)$, then $U(x,y) = x \vee y = y$. Since $b^y > g(x)$ and $b^y \in K$, using commutativity of U on K, $U(b^y,x) = U(x,b^y) = x \vee b^y = b^y$. Thus, $x \geq \widetilde{g}(b^y)$. By definition of the function $\widetilde{g}$, $\widetilde{g}(y) = \widetilde{g}(b^y)$ and so, $x \geq \widetilde{g}(y)$. Hence, we have $U(y,x) = y \vee x = y$ and so, $U(x,y) = U(y,x)$.

2. $x \notin K$ and $y \in K$,

2.1. $x < \widetilde{g}(y)$, then $U(y,x) = x \wedge y = x$. Since $b_x < \widetilde{g}(y)$, $U(y,b_x) = y \wedge b_x = b_x$. Since $b_x \in K$, using commutativity of U on K, $U(b_x,y) = U(y,b_x) = b_x$. Thus, $y \leq \widetilde{g}(b_x)$. By definition of the function of $\widetilde{g}$, $\widetilde{g}(x) = \widetilde{g}(b_x)$ and so, $y \leq \widetilde{g}(x)$. Hence, we have $U(x,y) = x \wedge y = x$ and so, $U(x,y) = U(y,x)$.

2.2. $x \geq \widetilde{g}(y)$, then $U(y,x) = y \vee x = y$. Since $b_x \geq \widetilde{g}(y)$, $U(y,b_x) = y \vee b_x = y$. Since $b_x \in K$, using commutativity of U on K, $U(b_x,y) = U(y,b_x) = y$. By using the monotonicity of U, $U(x,y) \geq U(b_x,y) = y$. Since $U(x,y) \in \{y,x\}$, we have $U(x,y) = y$ and so, $U(x,y) = U(y,x)$.

3. $x \notin K$ and $y \notin K$,

3.1. $y \leq \widetilde{g}(x)$, then $U(x,y) = x \wedge y = x$. Since $b^y \leq \widetilde{g}(x)$ and $b^y \in K$, using commutativity of U (case 2) $U(b^y,x) = U(x,b^y) = x \wedge b^y = x$. By the monotonicity of U, $U(y,x) \leq U(b^y,x) = x$. Since $U(y,x) \in \{y,x\}$, we have $U(y,x) = x$ and so, $U(x,y) = U(y,x)$.

3.2. $y > g(x)$, then $U(x,y) = x \vee y = y$. Since $b^y > \widetilde{g}(x)$ and $b^y \in K$, using commutativity of U (case 2), $U(b^y,x) = U(x,b^y) = x \vee b^y = b^y$. Thus, $x \geq \widetilde{g}(b^y)$. By definition of the function $\widetilde{g}$, $\widetilde{g}(y) = \widetilde{g}(b^y)$ and so, $x \geq \widetilde{g}(y)$. Hence, we have $U(y,x) = y \vee x = y$ and so, $U(x,y) = U(y,x)$.

iii) Neutral element: We show that $U(x,e) = x$ for all $x \in L$.

1. Let $x \leq e$. Then $U(x,e) = x \wedge e = x$.

2. Let $x > e$. Then $U(x,e) = x \vee e = x$.

iv) Associativity: We demonstrate that $U(x,U(y,z) = U(U(x,y),z)$ for all $x,y,z \in L$. Again the proof is split into all possible cases considering the relationships of the elements x,y,z and e.

U is idempotent, commutative and increasing and has the neutral element e. Due to Remark 2.3, U is associative for elements of L such that they are comparable with each elements of L. For this reason, we only discuss elements which are incomparable with some elements of L.

1. Let $x \leq e$.

1.1.$y \leq e$,

1.1.1. $z \leq e$,

$$\begin{aligned} U(x,U(y,z)) &= U(x,y \wedge z) = x \wedge y \wedge z \\ &= U(x \wedge y,z) \\ &= U(U(x,y),z). \end{aligned}$$

1.1.2. $z > e$, the condition that $x \leq y$ and $x > y$ is clear due to Remark 2.3. Consider $x \parallel y$. Then $b_x = b_y$ and we have $\widetilde{g}(x \wedge y) = \widetilde{g}(x) = \widetilde{g}(y) = \widetilde{g}(b_x)$ by definition of the function $\widetilde{g}$. If $z \leq \widetilde{g}(x)$,

$$U(x,U(y,z)) = U(x,y \wedge z) = U(x,y) = x \wedge y$$

and

$$U(U(x,y),z) = U(x \wedge y,z) = x \wedge y \wedge z = x \wedge y.$$

If $z > g(x)$,

$$U(x,U(y,z)) = U(x,y \vee z) = U(x,z) = x \vee z = z$$

and

$$U(U(x,y),z) = U(x \wedge y,z) = (x \wedge y) \vee z = z.$$

1.2. $y > e$,

1.2.1. $z \leq e$, the condition that $x \leq z$ and $x > z$ is clear due to Remark 2.3. Consider $x \parallel z$. Then $b_x = b_z$ and we have $\widetilde{g}(x \wedge z) = \widetilde{g}(x) = \widetilde{g}(z) = \widetilde{g}(b_x)$ by definition of the function $\widetilde{g}$. If $y \leq \widetilde{g}(x)$,

$$U(x,U(y,z)) = U(x,U(z,y)) = U(x,z \wedge y) = U(x,z) = x \wedge z$$

and

$$U(U(x,y),z) = U(x \wedge y,z) = U(x,z) = x \wedge z.$$

If $y > \widetilde{g}(x)$,

$$U(x,U(y,z)) = U(x,U(z,y)) = U(x,z \vee y) = U(x,y) = x \vee y = y$$

and

$$U(U(x,y),z) = U(x \vee y,z) = U(y,z) = U(z,y) = z \vee y = y$$

1.2.2. $z > e$, the condition that $y \leq z$ and $y > z$ is clear due to Remark 2.3. Consider $y \parallel z$. Then $b^y = b^z$ and we have $\widetilde{g}(y \vee z) = \widetilde{g}(y) = \widetilde{g}(z) = \widetilde{g}(b^y)$ by definition of the function $\widetilde{g}$. If $x < \widetilde{g}(y)$,

$$U(x,U(y,z)) = U(x,y \vee z) = U(y \vee z,x) = (y \vee z) \wedge x = x$$

and

$$U(U(x,y),z) = U(U(y,x),z) = U(y \wedge x,z) = U(x,z) = U(z,x) = x \wedge z = x.$$

If $x \geq \widetilde{g}(y)$,

$$U(x,U(y,z)) = U(x,y \vee z) = U(y \vee z,x) = y \vee z \vee x = y \vee z$$

and

$$U(U(x,y),z) = U(U(y,x),z) = U(y \vee x,z) = U(y,z) = y \vee z.$$

2. Let $x > e$.

2.1. $y \leq e$,

2.1.1. $z \leq e$, the condition that $y \leq z$ and $y > z$ is clear due to Remark 2.3. Consider $y \parallel z$. Then $b_y = b_z$ and we have $\widetilde{g}(y \wedge z) = \widetilde{g}(y) = \widetilde{g}(z) = \widetilde{g}(b_y)$ by definition of the function $\widetilde{g}$. If $x \leq \widetilde{g}(y)$,

$$U(x,U(y,z)) = U(x,y \wedge z) = U(y \wedge z,x) = (y \wedge z) \wedge x = y \wedge z$$

and

$$U(U(x,y),z) = U(U(y,x),z) = U(y \wedge x,z) = U(y,z) = y \wedge z.$$

If $x > \widetilde{g}(y)$,

$$U(x,U(y,z)) = U(x,y \wedge z) = U(y \wedge z,x) = (y \wedge z) \vee x = x$$

and

$$U(U(x,y),z) = U(U(y,x),z) = U(y \vee x, z) = U(x,z) = U(z,x) = z \vee x = x.$$

2.1.2. $z > e$, the condition that $x \leq z$ and $x > z$ is clear due to Remark 2.3. Consider $x \parallel z$. Then $b^x = b^z$ and we have $\widetilde{g}(x \vee z) = \widetilde{g}(x) = \widetilde{g}(z) = \widetilde{g}(b^x)$ by definition of the function $\widetilde{g}$. If $y < \widetilde{g}(x)$,

$$U(x, U(y,z)) = U(x, U(z,y)) = U(x, z \wedge y) = U(x,y) = x \wedge y = y$$

and

$$U(U(x,y),z) = U(x \wedge y, z) = U(y,z) = U(z,y) = z \wedge y = y.$$

If $y \geq \widetilde{g}(x)$,

$$U(x, U(y,z)) = U(x, U(z,y)) = U(x, z \vee y) = U(x,z) = x \vee z$$

and

$$U(U(x,y),z) = U(x \vee y, z) = U(x,z) = x \vee z.$$

2.2. $y > e$,

2.2.1. $z \leq e$, the condition that $x \leq y$ and $x > y$ is clear due to Remark 2.3. Consider $x \parallel y$. Then $b^x = b^y$ and we have $\widetilde{g}(x \vee y) = \widetilde{g}(x) = \widetilde{g}(y) = \widetilde{g}(b^x)$ by definition of the function $\widetilde{g}$. If $z < \widetilde{g}(x)$,

$$U(x, U(y,z)) = U(x, y \wedge z) = U(x,z) = x \wedge z = z$$

and

$$U(U(x,y),z) = U(x \vee y, z) = (x \vee y) \wedge z = z.$$

If $z \geq \widetilde{g}(x)$,

$$U(x, U(y,z)) = U(x, y \vee z) = U(x,y) = x \vee y$$

and

$$U(U(x,y),z) = U(x \vee y, z) = (x \vee y) \vee z = x \vee y.$$

2.2.2. $z > e$,

$$\begin{aligned} U(x, U(y,z)) &= U(x, y \vee z) = x \vee y \vee z \\ &= U(x \vee y, z) \\ &= U(U(x,y),z). \end{aligned}$$

□

As a consequence of the above consideration we have the following result.

Theorem 2.16. *Let $(L, \leq, 0, 1)$ be a bounded lattice, all $x \in L$ be comparable with e $(0 < e < 1)$, $Card(L) < \aleph_0$ and K be defined by* (2.15). *Operation $U : L \to L$ is an idempotent uninorm with the neutral element e if and only if there exists a decreasing function $g : K|_{[0,e]} \to K|_{[e,1]}$ with $g(e) = e$, such that U is given by* (2.16).

Example 2.8. a) Let us consider the uninorm given in Example 2.7. Then $K = \{0,a,b,r,q,e,s,t,m,p,1\}$ and the decreasing function $g : K\mid_{[0,e]} \to K\mid_{[e,1]}$ is given as

$$g(x) = \begin{cases} e & x \in \{e,q\}, \\ s & x \in \{b,r\}, \\ m & x = a, \\ 1 & x = 0. \end{cases}$$

Its symmetrical extension is given by

$$\bar{g}(x) = \begin{cases} 0 & x \in \{p,1\}, \\ a & x \in \{t,m\}, \\ r & x \in \{s,z\}, \\ e & x \in \{e,q\}, \\ s & x \in \{b,r\}, \\ m & x = a, \\ 1 & x = 0 \end{cases}$$

and $g^{\vee}$ by

$$g^{\vee}(x) = \begin{cases} a & x \in \{p,1\}, \\ b & x \in \{t,m\}, \\ q & x \in \{s,z\}, \\ e & x \in \{e\} \end{cases}$$

and finally, the function $\widetilde{g}$ on L is defined as

$$\widetilde{g}(x) = \begin{cases} a & x \in \{p,1\}, \\ b & x \in \{t,l,h,m\}, \\ q & x \in \{s,z\}, \\ e & x \in \{e,q\}, \\ s & x \in \{b,c,d,r\}, \\ m & x = a, \\ 1 & x = 0. \end{cases}$$

b) Let us now consider the function $g : K\mid_{[0,e]} \to K\mid_{[e,1]}$ as follows

$$g(x) = \begin{cases} p & x = 0, \\ m & x = a, \\ t & x = b, \\ s & x \in \{r,q\}, \\ e & x = e, \end{cases}$$

then its symmetrical extension is given by

$$\bar{g}(x)=\begin{cases} p & x=0,\\ m & x=a,\\ t & x=b,\\ s & x\in\{r,q\},\\ e & x=e,\\ q & x\in\{z,s\},\\ b & x=t,\\ a & x=m,\\ 0 & x\in\{p,1\} \end{cases}$$

and the function $\widetilde{g}$ is defined as

$$\widetilde{g}(x)=\begin{cases} p & x=0,\\ m & x=a,\\ t & x\in\{b,c,d\},\\ s & x\in\{r,q\},\\ e & x\in\{e,z,s\},\\ r & x=t,\\ b & x\in\{m,l,h\},\\ a & x=p,\\ 0 & x=1. \end{cases}$$

Using the formula (2.16) we obtain the uninorm given in Table 2.4.

Question 2.2. Is it possible to replace the condition that all elements are comparable with the neutral element by another condition, in such a way to obtain locally internal operation in the extended sense.

Question 2.3. Can we find two functions g_1, g_2, such that $g_1 \leq g_2$ and on $A(e)$ we have

$$U(x,y)=\begin{cases} x\wedge y \; if \; y<g_1(x),\\ x\vee y \; if \; y>g_2(x),\\ ||e \quad if \; g_1(x)<y<g_2(x). \end{cases}$$

2.5 Uninorms on interval-valued fuzzy sets

A special class of bounded lattices is the lattice of intervals associated with interval-valued fuzzy sets. Here we will present basic information about uninorms on the

U	0	a	b	c	d	r	q	e	z	s	t	l	h	m	p	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
a	0	a	a	a	a	a	a	a	a	a	a	a	a	a	p	1
b	0	a	b	b	b	b	b	b	b	b	b	l	h	m	p	1
c	0	a	b	c	b	c	c	c	c	c	c	l	h	m	p	1
d	0	a	b	b	d	d	d	d	d	d	d	l	h	m	p	1
r	0	a	b	c	d	r	r	r	r	r	t	l	h	m	p	1
q	0	a	b	c	d	r	q	q	q	q	t	l	h	m	p	1
e	0	a	b	c	d	r	q	e	z	s	t	l	h	m	p	1
z	0	a	b	c	d	r	q	z	z	s	t	l	h	m	p	1
s	0	a	b	c	d	r	q	s	s	s	t	l	h	m	p	1
t	0	a	b	c	d	t	t	t	t	t	t	l	h	m	p	1
l	0	a	l	l	l	l	l	l	l	l	l	l	m	m	p	1
h	0	a	h	h	h	h	h	h	h	h	h	m	h	m	p	1
m	0	a	m	m	m	m	m	m	m	m	m	m	m	m	p	1
p	0	p	p	p	p	p	p	p	p	p	p	p	p	p	p	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 2.4 The idempotent uninorm given in Example 2.8

lattice of intervals L^I, where

$$L^I = \{x = [x_1, x_2] : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \le x_2\}$$

with the natural order $\le_{L^I}$ defined as follows

$$[x_1, x_2] \le_{L^I} [y_1, y_2] \Leftrightarrow x_1 \le y_1,\ x_2 \le y_2,$$

and the greatest element $1 = [1,1]$ and the least element $0 = [0,0]$.

Definition 2.12 (cf. [69]). Operation $U : (L^I)^2 \to L^I$ is called a uninorm if it is commutative, associative, increasing and has the neutral element $e \in L^I$. If $e = 1$ then it is called a t-norm and if $e = 0$ it is called a t-conorm.

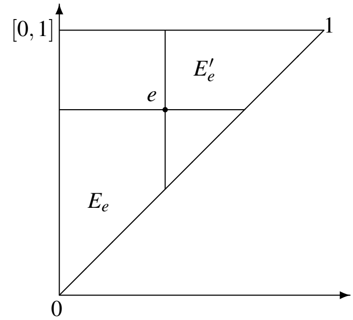

Fig. 2.7 The lattice of intervals L^I

Let us denote

$$E_e = \{x \in L^I : x \leq e\},$$
$$E'_e = \{x \in L^I : x \geq e\},$$
$$D = \{[x,x] : x \in [0,1]\}.$$

If we want to get a similar description as in (1.12) then we have to make the appropriate assumptions because of the following result.

Theorem 2.17 (cf. [69]). *Let $e \in L^I \setminus \{0,1\}$. If $e \notin D$, then it does not exist an increasing bijection $\Phi_e : L^I \to E_e$ such that Φ_e^{-1} is increasing and it does not exist an increasing bijection $\Psi_e : L^I \to E'_e$ such that Ψ_e^{-1} is increasing.*

Theorem 2.18 (cf. [68]). *If a uninorm U has the neutral element $e \in D \setminus \{0,1\}$, then there exist a t-norm T and a t-conorm S such that*

$$U(x,y) = \begin{cases} T^*(x,y) & \text{if } x,y \leq e, \\ S^*(x,y) & \text{if } x,y \geq e, \end{cases}$$

where

$$T^*(x,y) = \Phi_e^{-1}\left(T\left(\Phi_e(x), \Phi_e(y)\right)\right),\ \Phi_e(x) = (e_1x_1, e_1x_2),\ x,y \in E_e$$
$$S^*(x,y) = \Psi_e^{-1}\left(S\left(\Psi_e(x), \Psi_e(y)\right)\right),\ \Psi_e(x) = (e_1 + x_1 - e_1x_1, e_1 + (1-e_1)x_2) \text{ in } E'_e.$$

Lemma 2.10. *If U is a uninorm with the neutral element $e \in L^I$ then for all $x,y \in L^I$ such that $x \leq e \leq y$ we have*

$$x \leq U(x,y) \leq y.$$

Lemma 2.11 (cf. [80]). *If U is a uninorm with the neutral element $e \in L^I$ then for all $x,y \in L^I$ such that $x \leq e \leq y$ or $y \leq e \leq x$ we have*

$$\min(x,y) \leq U(x,y) \leq \max(x,y).$$

Moreover we have the following property.

Lemma 2.12. *Let U be a uninorm with the neutral element $e \in L^I \setminus \{0,1\}$. Then it holds:*

(a) $U(x,y) \leq x$ *for* $x \in L^I, y \in E$,
(b) $U(x,y) \leq y$ *for* $x \in E, y \in L^I$,
(c) $U(x,y) \geq x$ *for* $x \in L^I, y \in E$,
(d) $U(x,y) \geq y$ *for* $x \in E, y \in L^I$.

Question 2.4. Whether the neutral element must belong to the diagonal?

Definition 2.13 (cf. [68]). A uninorm $U : (L^I)^2 \to L^I$ is called t-representable if there exist uninorms $U_1, U_2 : [0,1]^2 \to [0,1]$ such that for all $x,y \in L^I$

$$U(x,y) = [U_1(x_1,y_1), U_2(x_2,y_2)].$$

Theorem 2.19. *If U is a t-representable uninorm with the neutral element $e = [e_1, e_2]$, then $e \in D$.*

Lemma 2.13 (cf. [69]). *If U is a uninorm with the neutral element $e \in L^I \setminus \{0,1\}$ then for all $x \in L^I$ we have*

$$U(0,1) = U(U(0,1),x).$$

Lemma 2.14 (cf. [69]). *If U is a uninorm with the neutral element $e \in L^I \setminus \{0,1\}$ then $U(0,1) = 0$ or $U(0,1) = 1$ or $U(0,1) \| e$.*

Example 2.9. Let U_{e_1} be the uninorm on $[0,1]$ given by

$$U_{e_1}(x,y) = \begin{cases} \max(x,y) & \text{if } x,y \in [e_1,1], \\ \min(x,y) & \text{else,} \end{cases}$$

then for the uninorm

$$U(x,y) = [U_{e_1}(x_1,y_1), U_{e_1}(x_2,y_2)]$$

on L^I we have $U(0,1) = [0,1]$ and U is neither conjunctive nor disjunctive.

Lemma 2.15. *If U is a t-representable uninorm with the neutral element $e \in L^I$ then $U(0,1) = 0$ or $U(0,1) = 1$ or $U(0,1) = [0,1]$.*

Question 2.5. Does the arbitrary uninorm U with the neutral element $e \in L^I$ satisfy condition
$U(0,1) = 0$ or $U(0,1) = 1$ or $U(0,1) = [0,1]$?

Question 2.6. Let $e \in L^I \setminus \{0,1\}$ and $z \in L^I$, such that $z \| e$ be fixed. Does it exist a uninorm U with the neutral element e and zero element z?

If neutral element is from the set D, then Deschrijver [68] presents the construction of uninorms that positive answer the above questions.

Theorem 2.20 (cf. [68]). *Let $e = [e_1, e_1] \in D \setminus \{0,1\}$, $z = [z_1, z_2] \in L^I$, T_1 and T_2 be t-norms, S_1 and S_2 be t-conorms on $[0,1]$ such that*

(i) $z \| e$,
(ii) there exist t-norms T_{1a} and T_{1b} such that T_1 is an ordinal sum of T_{1a} and T_{1b} with intervals $[0, \varphi(z_1)]$, $[\varphi(z_1), 1]$,
(iii) there exist t-conorms S_{1a} and S_{1b} such that S_1 is an ordinal sum of S_{1a} and S_{1b} with intervals $[0, \psi(z_2)]$, $[\psi(z_2), 1]$,
(iv) $T_1(x_1,y_1) \le T_2(x_1,y_1)$ and $S_1(x_1,y_1) \le S_2(x_1,y_1)$, for all $x_1, y_1 \in [0,1]$.

$$(U(x,y))_1 = \begin{cases} z_1 & if\ x_1 < z_1 \ and\ y_1 \ge z_1 \ and\ y_2 > e_1 \\ & or\ y_1 < z_1 \ and\ x_1 \ge z_1 \ and\ x_2 > e_1, \\ \varphi^{-1}\left(T_1\left(\varphi(x_1), \varphi(y_1)\right)\right) & if\ \max(x_1,y_1) \le e_1, \\ \psi^{-1}\left(S_1\left(\psi(x_1), \psi(y_1)\right)\right) & if\ \min(x_1,y_1) \ge e_1, \\ \min(x_1,y_1) & else, \end{cases}$$

$$(U(x,y))_2 = \begin{cases} z_2 & if\ (x_2 > z_2\ and\ y_2 \le z_2\ and\ y_1 < e_1 \\ & or\ (y_2 > z_2\ and\ x_2 \le z_2\ and\ x_1 < e_1, \\ \varphi^{-1}\left(T_2\left(\varphi(x_2), \varphi(y_2)\right)\right) & if\ \max(x_2,y_2) \le e_1, \\ \psi^{-1}\left(S_2\left(\psi(x_2), \psi(y_2)\right)\right) & if\ \min(x_2,y_2) \ge e_1, \\ \max(x_2,y_2) & else. \end{cases}$$

Then U is a uninorm on L^I with the neutral element e for which $U(0,1) = z$.

Another way to construct uninorms on a lattice of intervals is to use a construction on an arbitrary lattice. Sometimes the construction is too general and doesn't lead to a new class of uninorms, but sometimes we get uninorms that we didn't get from interval calculus by imposing too many restrictions.

Theorem 2.21 (cf. [137]). *Let $e \in L^I \setminus \{0,1\}$. Then U given by*

$$U([x_1,x_2],[y_1,y_2]) = \begin{cases} [x_1 \wedge y_1, x_2 \wedge y_2] & if\ x_1,y_1 \le e_1,\ x_2,y_2 \le e_2, \\ [x_1 \vee y_1, x_2 \vee y_2] & if\ (x_1 \le e_1 \le y_1\ and\ x_2 \le e_2 \le y_2) \\ & or\ (y_1 \le e_1 \le x_1\ and\ y_2 \le e_2 \le x_2), \\ [y_1,y_2] & if\ x_1 \le e_1, x_2 \le e_2\ and\ [y_1,y_2] \parallel [e_1,e_2], \\ [x_1,x_2] & if\ y_1 \le e_1, y_2 \le e_2\ and\ [x_1,x_2] \parallel [e_1,e_2], \\ [1,1] & otherwise \end{cases}$$

is a uninorm with the neutral element e.

2.6 Discrete uninorms

Having a finite number of alternatives that can be ordered linearly and whose values are, for example, in the unit range, we need aggregations on such a set. Since the essence of the set is not important here, we can use another model, i.e., we may number the alternatives from 0 to n and consider aggregations on such a set. Both of these approaches can be found in the literature, as well as operations on such sets (see [192, 234]).

Thus, given any positive integer n, we will deal from now on with the finite chain $L_n = \{0,1,2,...,n\}$ or equivalently using the interval notation $L_n = [0,n]$.

Definition 2.14 (cf. [177]). A uninorm on L_n is a function $U : L_n^2 \to L_n$ which is associative, commutative, increasing in each variable and such that there exists some element $e \in L_n$, called neutral element, such that $U(e,x) = x$ for all $x \in L_n$.

If $e = n$ then the function U becomes a t-norm and when $e = 0$ U becomes a t-conorm. Let us denote $E = [0,e) \times (e,n] \cup (e,n] \times [0,e)$.

Theorem 2.22. *Let $U : L_n^2 \to L_n$ be a uninorm with the neutral element $e \in L_n$. Then there exist a t-norm T on the interval $[0,e]$, and a t-conorm S on the interval $[e,n]$ such that*

$$U(x,y) = \begin{cases} T(x,y) & \text{if } x,y \in [0,e], \\ S(x,y) & \text{if } x,y \in [e,1]. \end{cases}$$

Moreover, $\min(x,y) \leq U(x,y) \leq \max(x,y)$ *for all* $(x,y) \in E$ *and* $U(0,n) \in \{0,n\}$. *A uninorm* U *is called conjunctive in the case when* $U(n,0) = 0$ *and disjunctive in the case when* $U(n,0) = n$.

Example 2.10. These are the basic triangular norms and their dual triangular conorms defined for $x,y \in L_n$:

- minimum t-norm and maximum t-conorm

$$T_M(x,y) = \min(x,y), \qquad S_M(x,y) = \max(x,y),$$

- Łukasiewicz t-norm and t-conorm

$$T_L(x,y) = \max(x+y-n,0), \qquad S_L(x,y) = \min(x+y,n),$$

- drastic t-norm and t-conorm

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = n, \\ 0 & \text{otherwise,} \end{cases}$$
$$S_D(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Instead of the continuity at L_n we will consider the smoothness property.

Definition 2.15 (cf. [192]). A function $f : L_n \to L_n$ is said to be smooth whenever $|f(x) - f(x-1)| \leq 1$ for all $x \in L_n$ such that $x \geq 1$.

Definition 2.16 (cf. [192]). A binary operation $F : L_n^2 \to L_n$ is said to be smooth when its vertical and horizontal sections, $F(x,\cdot)$ and $F(\cdot,y)$ are smooth.

The Archimedean t-norm ($T(x,x) < x$ for all $x \in L_n \setminus \{0,n\}$) are described as follows.

Theorem 2.23 (cf. [192, 234]). *The only Archimedean smooth t-norm and t-conorm on* L_n *are, respectively, the Łukasiewicz t-norm* T_L *and the Łukasiewicz t-conorm* S_L.

Smooth t-norms were characterized in [191, 192], using ordinal sum construction.

Theorem 2.24 ([192]). *A t-norm* T *on* L_n *is smooth if and only if there exists a natural number* k *with* $0 \leq k \leq n-1$ *and a subset* I *of* L_n, $I = \{0 = a_0 < a_1 < \ldots < a_k < a_{k+1} = n\}$ *such that* T *is given by*

$$T(x,y) = \begin{cases} \max(a_i, x+y-a_{i+1}) & \text{if } (x,y) \in [a_i,a_{i+1}]^2 \text{ and } 0 \leq i \leq k, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

There exist dual results for t-conorms that states that they are the maximum, the Łukasiewicz or ordinal sums of them [192].

As in the case of uninorm on the lattice, attempts are made to obtain analogous classes for discrete uninorms as in the case of uninorm on the unit interval. Idempotent uninorms are described in Subsubsection 2.4.1.1. Here we will present only uninorms from the classes $\mathscr{U}_{\min}$ and $\mathscr{U}_{\max}$ and uninorms with the underlying Archimedean operators. Other uninorm classes can be found in [234, 175].

Definition 2.17 (cf. [177]). A binary operation $U : L_n^2 \to L_n$ is a uninorm in $\mathscr{U}_{\min}$ with the neutral element $0 < e < n$ whenever there are a t-norm T on $[0,e]$ and a t-conorm S on $[e,n]$ such that U is given by

$$U(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ S(x,y) & \text{if } (x,y) \in [e,n]^2, \\ \min(x,y) & \textit{elsewhere}. \end{cases}$$

A binary operation $U : L_n^2 \to L_n$ is a uninorm in $\mathscr{U}_{\max}$ with the neutral element $0 < e < n$ whenever there are a t-norm T on $[0,e]$ and a t-conorm S on $[e,n]$ such that U is given by

$$U(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ S(x,y) & \text{if } (x,y) \in [e,n]^2, \\ \max(x,y) & \textit{elsewhere}. \end{cases}$$

Note, that the only Archimedean smooth t-norm and t-conorm are the Łukasiewicz operators. So, we have the following results.

Theorem 2.25 ([234]). *Let U be a conjunctive uninorm on L_n with the neutral element $0 < e < n$. If $T = T_L$ then $U \in \mathscr{U}_{\min}$.*

Theorem 2.26 ([234]). *Let U be a disjunctive uninorm on L_n with the neutral element $0 < e < n$. If $S = S_L$ then $U \in \mathscr{U}_{\max}$.*

The general structure of a uninorm with the underlying t-norm and t-conorm given by the ordinal sum of Łukasiewicz operators is described as follows

Theorem 2.27 ([234]). *A binary operation $U : L_n^2 \to L_n$ with the neutral element $0 < e < n$ and idempotent elements $J = \{0 = a_0 < a_1 < \ldots < a_r = e = b_0 < b_1 < \ldots < b_s = n\}$ is a uninorm with the smooth underlying operators T and S if and only if there exists a decreasing function $g : [0,e] \to [e,n]$ with the fix point e that satisfies the following conditions:*

(i) for all $a_i \in J$ there exists $b_j \in J$ such that $g(a_i) = b_j$,
(ii) if $x \in [a_i, a_{i+1})$, then $g(x) = g(a_i)$,

and U is given by

$$U(x,y) = \begin{cases} T(x,y) & \textit{if } (x,y) \in [0,e]^2, \\ S(x,y) & \textit{if } (x,y) \in [e,n]^2, \\ \min(x,y) & \textit{if } (x,y) \in E,\ y \leq \overline{g}(x) \textit{ and } x \leq \overline{g}(0), \\ \max(x,y) & \textit{elsewhere} \end{cases}$$

where $\overline{g}$ *is the only symmetric extension of* g *defined by*

$$\overline{g}(x) = \begin{cases} g(x) & \textit{if } x \leq e, \\ \max\{z \in [0,e] : g(z) \geq x\} & \textit{if } e \leq x \leq g(0), \\ 0 & \textit{if } x \geq g(0). \end{cases}$$

More information about the uninorms of such types can be found in [234, 49].

Chapter 3
Generalization of uninorms

One should always generalize.

C.G. Jacobi

In this chapter, we will give a brief overview of generalizations of uninorms without describing them in detail. One of the generalizations is to weaken the condition of the neutral element by dividing unit interval into subintervals in which operation has a neutral element (different in each case). These are the so-called n-uninorms, a special case of which are nullnorms, or uni-nullnorms, null-uninorms, etc. Another generalization involves omitting the commutativity condition. We get pseudo uninorms here. It turns out, however, that in the case of joint operations, some classes of these operations are commutative, or have a finite number of points for which the operation is non-commutative. An example of operations for which the condition of the neutral element has been weakened and the commutativity has been omitted are pseudo t-operators (in the literature also known as semi t-operators).

3.1 Pseudo uninorms

Definition 3.1. Operation $U : [0,1]^2 \to [0,1]$ is called a pseudo uninorm if it is associative, increasing and has the neutral element $e \in [0,1]$.

Remark 3.1. If $e = 1$ then we obtain pseudo t-norm and if $e = 0$ then we obtain pseudo t-conorm.

Theorem 3.1. *Let U be a pseudo uninorm with neutral element $e \in (0,1)$. Then there exists a pseudo t-norm T and a pseudo t-conorm S such that U is given by*

$$U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2. \end{cases} \tag{3.1}$$

To describe some properties of pseudo uninorms we can use Lemma 1.4
More details can be found in [168, 207, 208]

3.2 n-uninorms

n-uninorms were introduced by Akella [5] in 2007 to generalize the concept of a neutral element.

Definition 3.2 (cf. [5]). Let $n \in \mathbb{N}$ be natural number, $U : [0,1]^2 \to [0,1]$ a binary operation. The set $\{e_1, e_2, ..., e_n\}_{z_1, z_2, ..., z_{n-1}}$ is called n-neutral element, where $0 = z_0 < z_1 < z_2 < \ldots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$ such that, $U(e_i, x) = U(x, e_i) = x$ for all $x \in [z_{i-1}, z_i]$ for $i = 1, 2, \ldots, n$.

Definition 3.3 (cf. [5]). Operation $U : [0,1]^2 \to [0,1]$ is called a n-uninorm if it is commutative, associative, increasing and has the n-neutral element.

As an example of n-uninorm ($n = 2$) we can give nullnorms.

Definition 3.4 ([34]). Operation $V : [0,1]^2 \to [0,1]$ is called nullnorm if it is commutative, associative, increasing, has a zero element $z \in [0,1]$, and satisfies

$$V(0,x) = x \quad \text{for all } x \leq z, \tag{3.2}$$

$$V(1,x) = x \quad \text{for all } x \geq z. \tag{3.3}$$

Theorem 3.2 ([34]). *Let $z \in (0,1)$. A binary operation V is a nullnorm with zero element z if and only if there exists triangular norm T and triangular conorm S such that*

$$V(x,y) = \begin{cases} zS(\frac{x}{z}, \frac{y}{z}) & \text{if } x,y \in [0,z], \\ z + (1-z)T(\frac{x-z}{1-z}, \frac{y-z}{1-z}) & \text{if } x,y \in [z,1] \\ z \quad otherwise. & \end{cases}, \tag{3.4}$$

Additional information on n-uninorms can be found in [6, 201, 283, 206].

3.3 Weak uninorm

Instead of assuming that there is a division of the unit interval into n subintervals in which the operation has a neutral element, we can assume that for each point there is a neutral element and we get the concept of weak uninorms.

Definition 3.5 (cf. [162]). Operation $U : [0,1]^2 \to [0,1]$ is called a weak uninorm if it is commutative, associative, increasing and for any $x \in [0,1]$ there exist some identity element $u_x \in [0,1]$ such that $U(x, u_x) = x$.

For more information about weak uninorms one can see [132].

Other generalizations are left and right uninorms that have a one-sided neutral element instead of a neutral element. In addition, we also drop the commutativity condition here. This means that they are generalizations of pseudo uninorms. For more details see [184, 185].

Part II
Applications

Chapter 4
Inverse fuzzy implications

One of the most popular approaches to knowledge representation are the fuzzy production rules. In such situation, they are often presented in the form of IF-THEN and interpreted as implications. They are employed in inference schemas like *modus ponens, modus tollens*, etc. There exist uncountably many implication functions in the field of fuzzy logic, and the nature of the fuzzy inference changes variously depending on the implication function to be used. The variety of implication functions existing in the fuzzy set framework has always been seen as a rich potential for modeling different shades of expert attitude in the inference process (e.g. [138]), although no precise, practical interpretation was provided for the different implication functions [193]. Moreover, it is very difficult to select a suitable implication function for actual applications. In the process of selecting the appropriate implication in forward and backward reasoning, it is important to construct inverse implications with respect to the antecedent and the conclusion. As shown in the papers [248, 249, 246], among the typical examples of fuzzy implications, there is the problem of inversibility over the entire unit square. Since the implication family forms a lattice, one way to solve the problem is to find the largest or smallest implication with respect to the inverse implication in a given subset (the partition depends on the selected family of implications) and combine them to obtain the optimal implication. In this chapter we will show that the (UN)-implication is invertible with respect to the antecedent as well as the conclusion, assuming that the uninorm is representable, and that the family of (UN)-implications form an ordered family with respect to the parameter e, with a fixed generator of a uninorm. Thus, we can more easily choose the appropriate implication for the problem under consideration.

4.1 Fuzzy implications

We recall here some facts according to the notion of a fuzzy implication.

Definition 4.1 ([17] p. 2, [101] p. 21). A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it is:

(I1) decreasing in its first variable,
(I2) increasing in its second variable,
and
(I3) $I(0,0) = 1$,
(I4) $I(1,1) = 1$,
(I5) $I(1,0) = 0$.

Example 4.1 (cf. [17] pp. 4,5,57). There are many known examples of fuzzy implications. We recall here only those used later.
Let us present a family of fuzzy implications with parameter $\alpha \in [0,1]$

$$I_\alpha(x,y) = \begin{cases} 0, & \text{if } x = 1, y = 0 \\ 1, & \text{if } x = 0 \text{ or } y = 1 \\ \alpha & \text{otherwise} \end{cases}.$$

The operations I_0 and I_1 are the least and the greatest fuzzy implication, respectively.

$$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{otherwise} \end{cases} \quad I_1(x,y) = \begin{cases} 0, & \text{if } x = 1, y = 0 \\ 1, & \text{otherwise} \end{cases}$$

Some other, well-known, examples of fuzzy implications are listed below.

$$I_{LK}(x,y) = \min(1-x+y,1) \quad I_{GG}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{x}, & \text{otherwise} \end{cases}$$

$$I_{GD}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{otherwise} \end{cases} \quad I_{RS}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{otherwise} \end{cases}$$

$$I_{RC}(x,y) = 1-x+xy \quad I_{YG}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{otherwise} \end{cases}$$

$$I_{KD}(x,y) = \max(1-x,y) \quad I_{FD}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(1-x,y), & \text{otherwise} \end{cases}$$

$$I_{WB}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{otherwise} \end{cases} \quad I_{DP}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 1-x, & \text{if } y = 0 \\ 1, & \text{otherwise} \end{cases}$$

$$I_D(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{otherwise} \end{cases} \quad I'_D(x,y) = \begin{cases} 1, & \text{if } y = 1 \\ 1-x, & \text{otherwise} \end{cases}$$

There are many potential properties of fuzzy implications (see, e.g. [17] p. 9, [238]). We list below those that are considered further in this chapter:

- the consequent boundary property

$$I(x,y) \ge y, \;\; x,y \in [0,1], \tag{CB}$$

- the neutral property

$$I(1,y) = y, \quad y \in [0,1], \tag{NP}$$

- the identity principle

$$I(x,x) = 1, \quad x \in [0,1], \tag{IP}$$

- the exchange principle

$$I(x, I(y,z)) = I(y, I(x,z)), \quad x,y,z \in [0,1], \tag{EP}$$

- the left ordering property

$$x \leqslant y \Rightarrow I(x,y) = 1, \quad x,y \in [0,1], \tag{LOP}$$

- the right ordering property

$$I(x,y) = 1 \Rightarrow x \leqslant y, \quad x,y \in [0,1], \tag{ROP}$$

- the ordering property

$$I(x,y) = 1 \Leftrightarrow x \leqslant y, \quad x,y \in [0,1], \tag{OP}$$

- the strong boundary condition

$$I(x,0) = 0, \quad x \in (0,1], \tag{SBC}$$

- the strong corner condition for 0

$$I(x,y) = 0 \Rightarrow x = 1 \text{ and } y = 0, \quad x,y \in [0,1], \tag{SCC0}$$

- the strong corner condition for 1

$$I(x,y) = 1 \Rightarrow x = 0 \text{ or } y = 1, \quad x,y \in [0,1]. \tag{SCC1}$$

Definition 4.2. Let I be a fuzzy implication. The function N_I defined by

$$N_I = I(x,0), \;\; x \in [0,1]$$

is called the natural negation of I.

Example 4.2. Below we list the natural negations of the basic fuzzy implications from Example 4.1:

$$\begin{array}{lll} N_{I_{LK}} = N_S & N_{I_{GD}} = N_0 & N_{I_{RC}} = N_S \\ N_{I_{DN}} = N_S & N_{I_{GG}} = N_0 & N_{I_{RS}} = N_0 \\ N_{I_{YG}} = N_0 & N_{I_{WB}} = N_1 & N_{I_{FD}} = N_S \end{array}$$

4.2 (*UN*)-implications

In order to be able to define a special family of inverse implications, we consider (*UN*)-implications for the particular class of a uninorm.

At the beginning we will recall the concept of (SN)-implication, and then its generalization will be given.

Definition 4.3 ([17]). A function $I : [0,1]^2 \to [0,1]$ is called an (S,N)-implication if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x),y), \quad x,y \in [0,1]. \tag{4.1}$$

If N is a strong fuzzy negation, then I is called a strong implication or S-implication. Moreover, if I is an (S,N)-implication generated from S and N, then we will denote it by $I_{S,N}$.

A natural generalization of (S,N)-implications in the uninorm framework is to consider a uninorm in the place of a t-conorm in Definition 4.3.

Definition 4.4 ([17]). A function $I : [0,1]^2 \to [0,1]$ is called a (U,N)-operation, if there exist a uninorm U and a fuzzy negation N such that

$$I(x,y) = U(N(x),y), \quad x,y \in [0,1]. \tag{4.2}$$

If I is a (U,N)-operation generated from a uninorm U and a negation N, then we will denote it by $I_{U,N}$.

Theorem 4.1 ([17]). *Let U be a uninorm with the neutral element $e \in (0,1)$ and N any fuzzy negation. Then the following statements are equivalent:*

(i) The (U,N)-operation $I_{U,N}$ is a fuzzy implication.
(ii) U is a disjunctive uninorm.

Example 4.3 (cf. [17] Example 5.3.6.). In the following, we give examples of (U,N)-implications obtained using the classical strong negation N_S and different uninorms.

(i) Let us consider a disjunctive uninorm $U_{L,L}^{\max}$ from the class $\mathscr{U}_{\max}$ generated by the triplet $(T_L, S_L, 0.5)$. Then

$$I_{U_{L,L}^{\max},N_S}(x,y) = \begin{cases} \max(y-x+0.5,0) & if \ \max(1-x,y) \leq 0.5, \\ \min(y-x+0.5,1) & if \ \max(1-x,y) > 0.5, \\ I_{KD}(x,y) & otherwise. \end{cases}$$

(ii) Let us consider the disjunctive representable uninorm U_u^d from Example 1.14 with the neutral element $e = 0.5$. Then

$$I_{U_u^d,N_S}(x,y) = \begin{cases} 1 & if \ (x,y) \in \{(0,0),(1,1)\}, \\ \frac{(1-x)y}{x+y-2xy} & otherwise. \end{cases}$$

(iii) Let now $U_{u,e}^d$ be the disjunctive representable uninorm with arbitrary, but fixed neutral element $e \in (0,1)$ (see Example 1.14). Then

$$I_{U^d_{u,e},N_S}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \{(0,0),(1,1)\}, \\ \frac{(1-e)(1-x)y}{(1-e)(1-x)y+ex(1-y)} & \text{otherwise.} \end{cases}$$

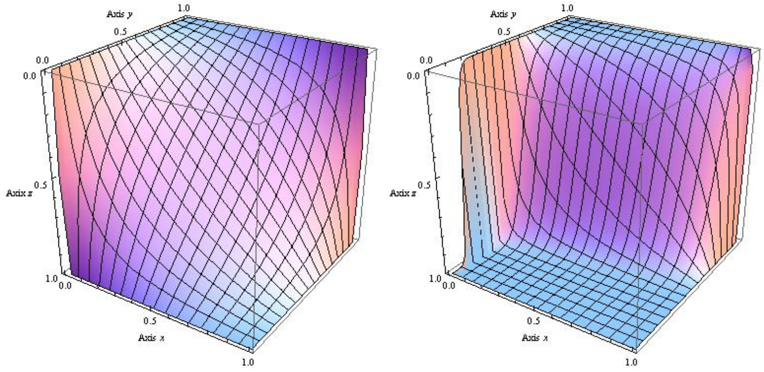

Fig. 4.1 Implication from Example 4.3 (ii) (left) and with using another negation than N_S ($N_{\underline{0.8}}$) (right)

Theorem 4.2. *Let us consider the implications* $I_{U^d_{u,e_1},N}$ *and* $I_{U^d_{u,e_2},N}$ *from Example 4.3 with the neutral element of uninorms* e_1 *and* e_2 *respectively. Then* $I_{U^d_{u,e_1},N} \leq I_{U^d_{u,e_2},N}$ *if* $e_1 \geq e_2$.

Proof. We get the thesis directly from the inequalities in Lemma 1.4 and Example 1.9.

Corollary 4.1. *Using Theorem 4.2 we can obtain a monotonic family of implications with respect to the parameter e.*

Remark 4.1. If we consider the fixed point *a* of a fuzzy negation *N* and the neutral element of a uninorm *U* then we can obtain two parameters *e* and *a* which can act as threshold parameters for the antecedent as well as for the consequences of the implication, or both.

Remark 4.2. I_{U,N_S} implication is symmetric about the line $y = 1 - x$ (because of the commutativity of the uninorm), so in a sense, increasing the antecedent by some value is equivalent to decreasing the consequent by the same value.
If we use a different negation, we can manipulate this dependency (see Figure 4.1, right part).

4.3 Constructing a fuzzy implication from given one(s)

In the next sections we recall chosen ideas of constructing a fuzzy implication from given one(s).

4.3.1 Threshold Generation Method

In 2012 Massanet and Torrens [189], generalizing the construction of h-implications from f- and g-implications, introduced the threshold generation method of a fuzzy implication from two given ones which was based on an adequate scaling on the second variable of the initial implications.

Definition 4.5 ([189]). Let I_1, I_2 be two fuzzy implications and $e \in (0,1)$. Then the function $I_{I_1-I_2} : [0,1]^2 \to [0,1]$, called the e-threshold generated implication from I_1 and I_2 is defined as

$$I_{I_1-I_2}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ e \cdot I_1\left(x, \frac{y}{e}\right) & \text{if } x > 0, y \le e, \\ e + (1-e) \cdot I_2\left(x, \frac{y-e}{1-e}\right) & \text{if } x > 0, y > e. \end{cases} \tag{4.3}$$

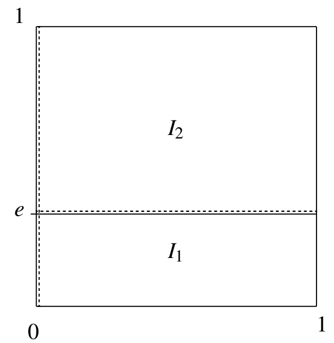

Fig. 4.2 Visualization of the structure of a fuzzy implication given by (4.3)

Theorem 4.3 ([189]). *The function* $I_{I_1-I_2} : [0,1]^2 \to [0,1]$, *given by* (4.3) *is a fuzzy implication.*

Example 4.4. Let $I_{U^d_{u,0.5},N_S}$ be the implication from Example 4.3, $e = \frac{1}{2}$. Using the formula (4.3) we obtain the following implication

$$I_{I_{U^d_{u,0.5},N_S}-I_{U^d_{u,0.5},N_S}}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{(1-x)y}{x+2y-4xy} & \text{if } x > 0,\ y \le \frac{1}{2}, \\ \frac{1}{2} + \frac{(1-x)(2y-1)}{2x+4y-2-4x(2y-1)} & \text{if } x > 0,\ y > \frac{1}{2}. \end{cases}$$

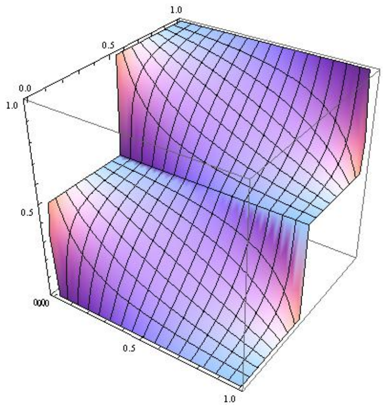

Fig. 4.3 Implication from Example 4.4

4.3.2 Multi-threshold Generation Method

Next, we present a construction, which is a kind of combination of the constructions from Definition 4.5, and ordinal sum construction. We consider a finite or countably infinite number of fuzzy implications as construction generators whose linearly transformed values in the appropriate areas are supplemented by I_D implication.

In the sequel, $|a,b|$ denotes one of the intervals (a,b), $(a,b]$, $[a,b)$ or $[a,b]$.

Definition 4.6 ([87]). Let $\{I_k\}_{k\in\mathscr{A}}$ be a family of fuzzy implications, $\{|a_k,b_k|\}_{k\in\mathscr{A}}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ for all $k \in \mathscr{A}$. The operation $I\colon [0,1]^2 \to [0,1]$ called multi-threshold fuzzy implication is defined by the following formula (see Figure 4.4)

$$I(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ a_k + (b_k - a_k)I_k\left(x, \frac{y-a_k}{b_k-a_k}\right) & \text{if } x > 0,\ y \in |a_k,b_k|, \\ y & \text{otherwise.} \end{cases} \tag{4.4}$$

Theorem 4.4 ([87]). *The function* $I : [0,1]^2 \to [0,1]$ *given by* (4.4) *is a fuzzy implication.*

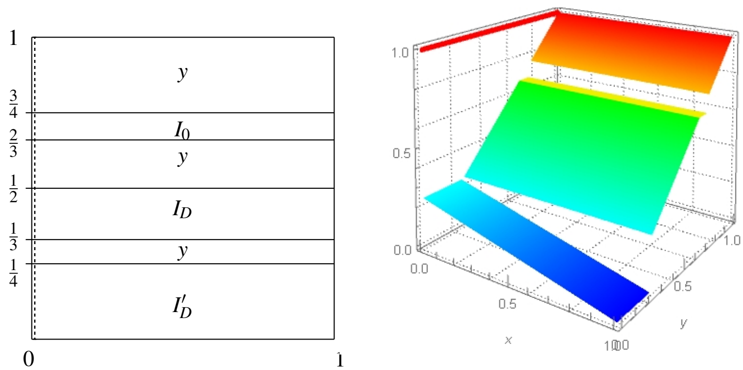

Fig. 4.4 2D and 3D-visualization of the structure of a fuzzy implication given by (4.4)

4.3.3 Vertical threshold generation method

In 2013 Massanet and Torrens [190], introduced the vertical threshold generation method of a fuzzy implication from two given ones which was based on an adequate scaling on the first variable of the initial implications.

Definition 4.7 ([190]). Let I_1, I_2 be two fuzzy implications and $e \in (0,1)$. Then the function $I_{I_1|I_2} : [0,1]^2 \to [0,1]$, called the vertical e-threshold generated implication from I_1 and I_2 is defined as

$$I_{I_1|I_2}(x,y) = \begin{cases} 1 & \text{if } y = 1, \\ e + (1-e) \cdot I_1\left(\frac{x}{e}, y\right) & \text{if } x \in [0,e),\ y \in [0,1), \\ e \cdot I_2\left(\frac{x-e}{1-e}, y\right) & \text{if } x \in [e,1],\ y \in [0,1). \end{cases} \tag{4.5}$$

Theorem 4.5 ([189]). *The function $I_{I_1|I_2} : [0,1]^2 \to [0,1]$, given by* (4.5) *is a fuzzy implication.*

Example 4.5. Let $I_{U^d_{u,0.5},N_S}$ be the implication from Example 4.3, $e = \frac{1}{2}$. Using the formula (4.5) we obtain the following implication (see Figure 4.5)

$$I_{I_{U^d_{u,0.5},N_S}|I_{U^d_{u,0.5},N_S}}(x,y) = \begin{cases} 1 & \text{if } y = 1, \\ \frac{1}{2} + \frac{(1-2x)y}{4x+2y-8xy} & \text{if } y < 1,\ x \geq \frac{1}{2}, \\ \frac{(1-x)y}{2x+y-1+(2-4x)y} & \text{if } y < 1,\ x < \frac{1}{2}. \end{cases}$$

4.3.4 Left ordinal sum of fuzzy implications

Here, we propose new definitions of ordinal sums of fuzzy implications which are modifications of the constructions recalled in Subsection 4.3.3.

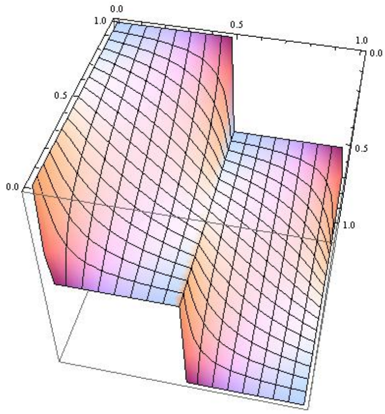

Fig. 4.5 Implication from Example 4.5

Definition 4.8 (cf. [85]). Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications, $\{[a_k,b_k]\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ for all $k \in A$, where A is a non-empty, finite or countably infinite index set. The operation $I\colon [0,1]^2 \to [0,1]$ defined by the following formula (see Figure 4.6)

$$I(x,y) = \begin{cases} (1-b_k)+(b_k-a_k)I_k\left(\frac{x-a_k}{b_k-a_k},y\right) & \text{if } x\in[a_k,b_k], y\in[0,1), \\ I'_D(x,y) & \text{otherwise} \end{cases} \tag{4.6}$$

is called the left ordinal sum of fuzzy implications $\{I_k\}_{k\in A}$.

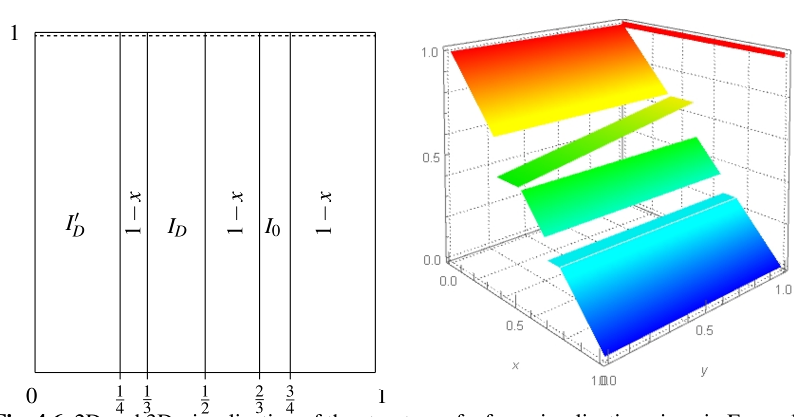

Fig. 4.6 2D and 3D-visualization of the structure of a fuzzy implication given in Example 4.6

Theorem 4.6 (cf. [85]). *The left ordinal sum of fuzzy implications given by* (4.6) *is a fuzzy implication.*

Example 4.6. Let us consider fuzzy implications $I_{k_1} = I'_D$, $I_{k_2} = I_D$, $I_{k_3} = I_0$ and a family of intervals $\{[0,\frac{1}{4}],[\frac{1}{3},\frac{1}{2}],[\frac{2}{3},\frac{3}{4}]\}$. The following operation $I:[0,1]^2 \to [0,1]$ is a fuzzy implication generated by the left ordinal sum (4.6) (see Figure 4.6)

$$I(x,y) = \begin{cases} 1 & if\ y=1, \\ \frac{1}{2}+\frac{1}{6}y & if\ x \in (\frac{1}{3},\frac{1}{2}],\ y \in [0,1), \\ \frac{1}{4} & if\ x \in (\frac{2}{3},\frac{3}{4}],\ y \in [0,1), \\ 1-x & \text{otherwise.} \end{cases}$$

4.4 Inverse implication with respect to antecedent

4.4.1 Definition and basic properties

Consider the basic fuzzy implication $I(x,y)$ where $x,y \in [0,1]$. Suppose the value of y is the truth value of the consequent and is known, the value $z = I(x,y)$ is the truth value of the implication and is also known. In order to determine the value of the truth of the implication's antecedent x the inverse function $InvI(y,z)$ has to be determined (see [248]). Note that in order to invert a function, it should be injective.

Example 4.7. Let us consider the implication I_{DN} (see Figure 4.7) from Example 4.1. Note that I_{DN} is not surjective. So, the domain of inverse fuzzy implication is the set $\{(y,z): y \le z < 1\ and\ y \in (0,1)\}$. The inverse fuzzy implication of I_{DN} is

$$InvI_{DN}(y,z) = 1-z.$$

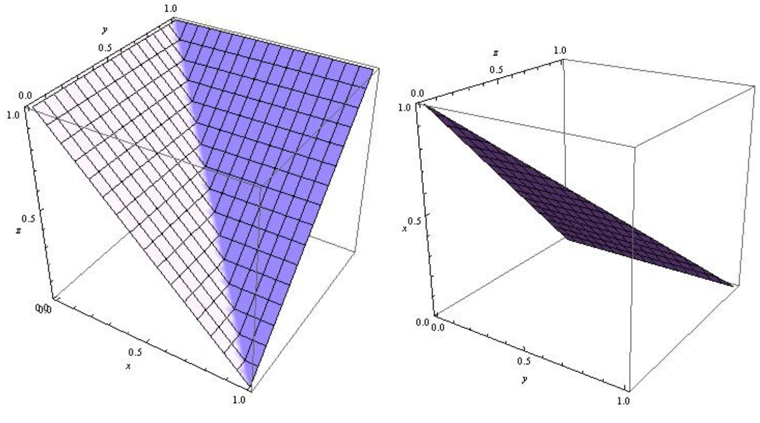

Fig. 4.7 Implication I_{DN} and its inverse implication $InvI_{DN}$

Example 4.7, as well as the examples presented in paper [248] show that the domain of the inverse implication is contained in the triangle above the main diagonal. This problem is explained in the following theorem.

Theorem 4.7 (cf. [86]). *If implication fulfills the condition* (NP) *then the domain of inverse fuzzy implication is included in a half of the unit square, where* $y \le z < 1$ *and* $y \in (0,1)$.

In the next example, we will show the implication for which the domain of the inverse implication is the whole unit square (excluding the boundary points, which will be discussed later).

Example 4.8. Let be given (U,N)-implication generated by the representable uninorm (see Example 4.3) with the neutral element $e = 0.5$ and standard negation

$$I_{U^d_{u,0.5},N_S}(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ \frac{(1-x)y}{(1-x)y+x(1-y)} & \text{otherwise.} \end{cases}$$

Then the inverse implication is given by the formula

$$InvI_{U^d_{u,0.5},N_S}(y,z) = \begin{cases} 1 & \text{if } y = 0,\ z = 0 \text{ or } y = 1,\ z = 1, \\ \frac{(1-z)y}{(1-z)y+z(1-y)} & \text{otherwise.} \end{cases}$$

Remark 4.3. Note that the inverse implications presented in Example 4.7 and 4.8 and examples from the paper [248] are all increasing for the first variable and decreasing for the second. Therefore, they do not meet the conditions set for the basic logical connectives. If, however, we consider the inverse of the fuzzy implication as a function of the variables z and y, then we obtain the monotonicity consistent with the monotonicity of the fuzzy implications. Other conditions for fuzzy implication will be also preserved, so we will adopt such a concept later in the book.

Remark 4.4. Note that the implication is constant for $x = 0$ and for $y = 1$. Therefore, taking Remark 4.3 into account, we complete the definition of the inverse implication on the boundary of the unit square. So, we assume that $InvI(0,x) = InvI(x,1) = 1$ for all $x \in [0,1]$.

Example 4.9. Let I be (U,N)-implication generated by the representable uninorm (see Example 4.3) with the neutral element e

$$I_{U^d_{u,e},N_S}(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ \frac{(1-e)(1-x)y}{(1-e)(1-x)y+ex(1-y)} & \text{otherwise.} \end{cases}$$

The inverse implication is given by the formula

$$InvI_{U^d_{u,e}N_S}(z,y) = \begin{cases} 1 & \text{if } y = 0,\ z = 0 \text{ or } y = 1,\ z = 1, \\ \frac{(1-e)(1-z)y}{(1-e)(1-z)y+ez(1-y)} & \text{otherwise.} \end{cases} \tag{4.7}$$

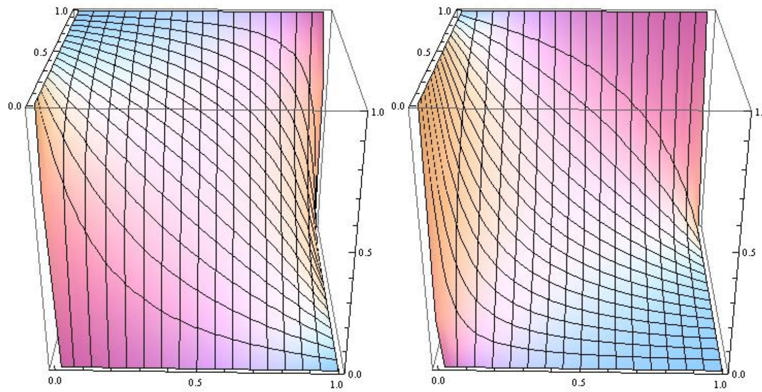

Fig. 4.8 Inverted implication for $e = 0.3$ and $e = 0.8$

Theorem 4.8 (cf. [86]). *Let $I_{U^d_{u,e},N_S}$ be (U,N)-implication generated by the representable uninorm (see Example 4.3) with the neutral element e. Then the inverse implication $InvI_{U^d_{u,e},N_S}$ given by* (4.7) *is a fuzzy implication.*

Theorem 4.9 (cf. [86]). *Let us consider the implications $I_{U^d_{u,e_1},N_S}$ and $I_{U^d_{u,e_2},N_S}$ from Example 4.3 with neutral element of uninorms e_1 and e_2 respectively. Then*

$$InvI_{U^d_{u,e_1},N_S} \le InvI_{U^d_{u,e_2},N_S} \text{ if } e_1 \ge e_2.$$

Corollary 4.2. *Using Theorem 4.9 we obtain the family $\{InvI_{U^d_{u,e},N_S}\}_{e\in(0,1)}$ which is a monotonic family of inverse implications with respect to the parameter e. Note that the family $\{I_{U^d_{u,e},N_S}\}_{e\in(0,1)}$ is also monotonic family of implications with respect to the parameter e.*

4.4.2 Threshold parameter

Sometimes, having a threshold parameter, we want the value of the truth of the implication's antecedent to be consistent with the truth value of the implication. We can get it using the vertical threshold generation method of fuzzy implications described in Subsection 4.3.3 or left ordinal sum of fuzzy implications described in Subsection 4.3.4. In such case, also the value of the truth of the implication's antecedent will be consistent with the truth value of the inverse implication.

Example 4.10. Let $I_{I_{U^d_{u,0.5},N_S}|I_{U^d_{u,0.5},N_S}}$ be the implication from Example 4.5. The inverse implication is given by the formula:

$$InvI_{I_{U^d_{u,0.5},N_S}|I_{U^d_{u,0.5},N_S}}(z,y) = \begin{cases} 1 & \text{if } y = 1, \\ \frac{1}{2} + \frac{(1-2z)y}{4z+2y-8zy} & \text{if } y < 1,\ z \geq \frac{1}{2}, \\ \frac{(1-z)y}{2z+y-1+(2-4z)y} & \text{if } y < 1,\ z < \frac{1}{2}. \end{cases}$$

4.4.3 Selection of implications

Another problem considered in the paper [248] is how to choose the appropriate function from the basic fuzzy implications. The first option is to select the implications with the largest or smallest values in a given subset (see also [86, 246]). But the authors proposed a different method of selecting the implications. Their method allows the comparison of two fuzzy implications. If the logical value of the consequent and the logical value of the implication are given by using the inverse fuzzy implication we can easily optimize the truth value of the implication's antecedent. In other words, we can choose the fuzzy implication that has the largest or smallest logical antecedent of the implication. The basic results for this problem consist in choosing an implication from a set of implications (e.g. implication from Example 4.1). If the choice is not satisfactory for us (some implications are incomparable), we divide the domain into several subsets (there are 19 in paper [248]) and on each subset we choose the one with the greatest inverse implication. Since, according to Theorem 4.9, the family of inverse implications from Example 4.9 is a linearly ordered set, we can select implications with the appropriate values (larger or smaller) by changing only the e parameter. This also solves the problem considered in paper [246], where looking for the optimal implication for given one, we get several chains, and for each of them we get the optimal implication. From the set of these implications, we should choose one, but this is not always possible (this is not considered in the paper [246]). However, considering the implication family from Example 4.3 we have a linear order with respect to the parameter e, so the problem is partially solved. In all considered problems, the use of a one-parameter family of implications (depending on e) will also reduce the computational complexity of determining the values of selected implication, for example, if we do no need to check whether a point belongs to one of the regions of the domain, as it was in paper [248].

4.5 Inverse implications with respect to consequent

4.5.1 Definition and basic properties

Another problem considered in the literature is finding the second type of inverse implication function (see [249, 86]). Given a basic fuzzy implication $I(x,y)$, where $x,y \in [0,1]$. The value x is the truth value of the implication's antecedent and is known. The value $z = I(x,y)$ is the truth value of the implication and is also known.

In order to determine the value of the truth of the consequent y the inverse function $Inv'I(x,z)$ has to be determined.
As in the previous section not every of basic implications can be inverted. The function can be inverted only when it is injective with respect to the consequent.

Example 4.11. Let us consider the implication I_{DN} (see Figure 4.7, left part) from Example 4.1 (see also Example 4.7). The domain of inverse fuzzy implication is the set $\{(x,z) : 1-x \leq z < 1 \textit{ and } x \in (0,1)\}$. The inverse fuzzy implication of I_{DN} is

$$Inv'I_{DN}(x,z) = z.$$

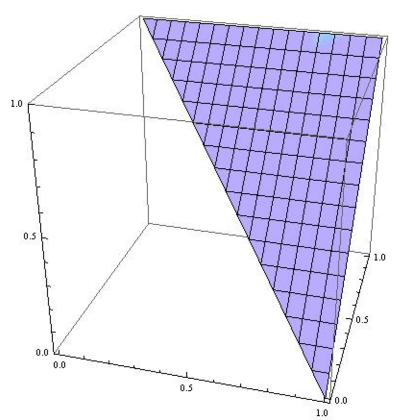

Fig. 4.9 Implication from Example 4.11

The domain of the inverse implication in Example 4.11 is the triangle above the diagonal. The general dependence concerning the domain of the inverse implication will be presented in the following theorem

Theorem 4.10 (cf. [86]). *The domain of inverse fuzzy implication is included in the region, for which $N_I(x) \leq z < 1$ and $x \in (0,1)$.*

Proof. According to the definition of N_I we have $N_I(x) = I(x,0) \leq I(x,y) < 1$, which gives the thesis.

It is easy to check that the following property hold.

Theorem 4.11. *If the domain of inverse fuzzy implication is $[0,1]^2$ then fuzzy implication fulfills the condition* (SBC).

Theorem 4.12. *If the implication fulfills the condition* (SCC0) *then the domain of inverse fuzzy implication is a proper subset of $(0,1)^2$.*

Example 4.12. a) The domain of inverse fuzzy implication given in Example 4.11 is a proper subset of $(0,1)^2$.

b) Let $U^d_{u,0.5}$ be the uninorm from Example 4.3 with $e = 0.5$. Then

$$U^d_{u,0.5}(x,y) = \begin{cases} 1 & \text{if } x = 0,\ y = 1 \text{ or } x = 1,\ y = 0, \\ \frac{xy}{xy+(1-x)(1-y)} & \text{otherwise} \end{cases}$$

and

$$I_{U^d_{u,0.5},N_S}(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \text{ or } x = y = 1, \\ \frac{(1-x)y}{(1-x)y+x(1-y)} & otherwise, \end{cases}$$

$$Inv' I_{U^d_{u,0.5},N_S}(x,z) = \begin{cases} 1 & \text{if } x = 0,\ z = 1 \text{ or } x = 1,\ z = 0, \\ \frac{xz}{(1-x)(1-z)+xz} & otherwise. \end{cases}$$

So, we can see, that $Inv' I_{U^d_{u,0.5},N_S} = U^d_{u,0.5}$. Moreover, the domain of inverse fuzzy implication is the whole unit square.

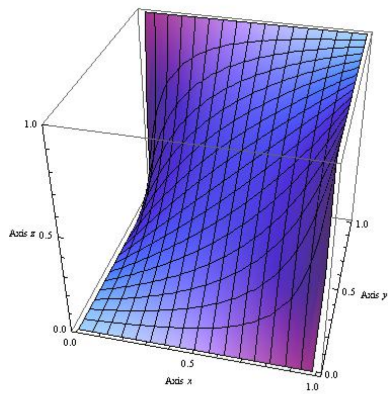

Fig. 4.10 $Inv' I$ from Example 4.12

Remark 4.5. Note that the inverse implication from Example 4.12 is increasing with respect to both variables. Moreover, implication $I_{U^d_{u,0.5},N_S}$ is constant on the boundary. Using the second line of the formula describing $I_{U^d_{u,0.5},N_S}$, we obtain the formula for the inverse fuzzy implication outside the corner points. At these points, we can insert any value in the $[0,1]$. In our considerations, we will assume the value of 1, so that the obtained result will be a disjunction.

Question 4.1. Are there any fuzzy implications that, when inverted with respect to the consequent, are the implications defined on the entire unit square?

More general examples of inverse implications are as presented below.

Example 4.13. Let consider the uninorm $U_{u,e}^{d}$ from Example 4.3 with fixed e. Then

$$I_{U_{u,e}^{d},N_S}(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ \frac{(1-e)(1-x)y}{(1-e)(1-x)y+ex(1-y)} & \text{otherwise,} \end{cases}$$

$$Inv'I_{U_{u,e}^{d},N_S}(x,z) = \begin{cases} 1 & \text{if } x = 0,\ z = 1 \text{ or } x = 1,\ z = 0, \\ \frac{exz}{(1-e)(1-x)(1-z)+exz} & \text{otherwise.} \end{cases}$$

Remark 4.6. Note that the inverse implications related to Example 4.12 and 4.13 and the paper [249] are increasing for both variables. Moreover, in the case of Example 4.12 and 4.13 we return to the uninorm which was used to build the implication. We get the same property for (S,N)-implications from the paper [249].

From the above we obtain the following property

Theorem 4.13. *Let $I_{U_{u,e}^{d},N_S}$ be the (U,N)-implication generated by the representable uninorm (see Example 4.3). Then the inverse implication $Inv'I_{U_{u,e}^{d},N_S}$ is a representable uninorm.*

Theorem 4.14. *Let us consider the implications I_{U_{u,e_1}^{d},N_S} and I_{U_{u,e_2}^{d},N_S} from Example 4.3 with the neutral element of uninorms e_1 and e_2 respectively. Then*

$$Inv'I_{U_{u,e_1}^{d}N_S} \leq Inv'I_{U_{u,e_2}^{d}N_S} \text{ if } e_1 \geq e_2.$$

Corollary 4.3. *Using Theorem 4.14 we can obtain the family $\{Inv'I_{U_{u,e}^{d},N_S}\}_{e\in(0,1)}$ which is a monotonic family of inverse implications with respect to the parameter e. Note that the family $\{I_{U_{u,e}^{d},N_S}\}_{e\in(0,1)}$ is also a monotonic family of implications with respect to the parameter e.*

Remark 4.7. Considering other families of representable uninorms (see Section 1.9) we can obtain various families of fuzzy implications and their inverses and create a lattice of them, which will allow us to select the appropriate implication for a particular application.

4.5.2 Threshold parameter

Sometimes, having a threshold parameter, we want the value of the truth of the consequent to be consistent with the truth value of the implication, i.e. both values are below the threshold parameter, or both are above the threshold parameter. We can get it using the threshold generation method of fuzzy implications or multi-threshold generation method of fuzzy implications described in Subsections 4.3.1 and 4.3.2.

Example 4.14. Let $I_{I_{U^d_{u,0.5},N_S}-I_{U^d_{u,0.5},N_S}}$ be the implication from Example 4.4. The inverse implication is given in the open unit square by the formula

$$Inv'I_{I_{U^d_{u,0.5},N_S}-I_{U^d_{u,0.5},N_S}}(x,z) = \begin{cases} \frac{xz}{1-x-2z-4xz} & \text{if } x>0, z\leq\frac{1}{2} \\ \frac{1}{2}+\frac{x(1-2z)}{6x-4+4z-8xz} & \text{if } x>0, z>\frac{1}{2}. \end{cases}$$

In fact, we get an operation that is built like an implication using the threshold generation method of fuzzy implications but with representable uninorms as components.

Chapter 5
Short notes on classifiers

Science is the systematic classification of experience.

G.H. Lewes

In the next chapters we will use the concept of a classifier and evaluate its quality. In order not to repeat the information in each of them, in this chapter we will present a very short introduction to classifiers and their evaluation. The main task of the classification constituting one of the important methods of data mining is the creation of models, called classifiers (also classifying algorithms or decision algorithms), describing dependencies between the given class (category) of objects and their characteristics. Discovered classification models are then used to classify new objects of the unknown class membership. In literature there can be found descriptions of numerous approaches to constructing classifiers, which are based on such paradigms of machine learning theory as classical and modern statistical methods, decision rules, decision trees, neural networks, and inductive logic programming (cf. [211, 19, 24, 20]). We will consider a problem of approximation of concepts (classes) based on a finite set of observations containing examples of positive and negative concepts. This finite set of observations may be represented using data tables. In this representation individual observations correspond to rows of a given data table and attributes to columns of a given data table. In this contribution, we consider decision tables of the form $\mathbf{T} = (U, A, d)$ in Pawlak's sense (cf. [219]) for representation of data tables, where U is a set of objects (rows) in the data table, A is a set of attributes or columns in the data table, and d is a distinguished attribute from the set A called a decision attribute (in this book, we consider problems for the case of a 2-class classification, e.g., for decision classes YES and NO or for decision classes 0 and 1, etc.).

object	age	blood pressure	d
u1	56	147	0
u2	71	154	1
u3	66	120	0
u4	64	131	0
u5	64	140	1

Table 5.1 Decision table

During classification, the classifier assigns a certain classification weight to the object (see Figure 5.1). For a set range of the threshold parameters $t \in (0,1)$, the test objects are tested in such a way that if the classification weight of the test object obtained from the classifier is greater than t, the object is classified into the main class (e.g., YES or 1). However, if the weight is less than or equal to t, then the object is classified into a subordinate class (e.g., NO or 0). In this way, we obtain the decision value for the test object, which may be correct (consistent with the actual decision in the test table) or incorrect (we made a mistake in the classification).

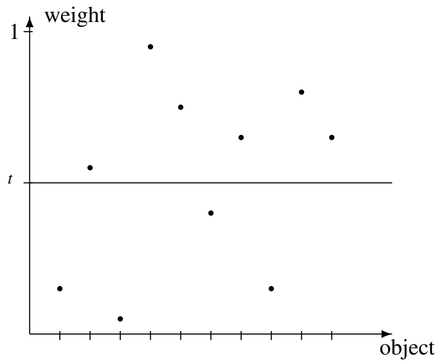

Fig. 5.1 Certain classification weights assigned by the classifier to objects and the threshold parameter $t \in (0,1)$

5.1 Quality measures of classifiers

To calculate the global classification quality of a given classifier with the fixed parameter t we use the accuracy (ACC) of the classification which is the quotient of the number of correct classifications to the number of all classifications.

Using the following notion:

- TP – True Positives – elements from the main class classified into the main class,
- TN – True Negatives – elements from the subordinate class classified into the subordinate class,
- FP – False Positives – elements from the subordinate class classified into the main class,
- FN – False Negatives – elements from the main class classified into the subordinate class,
- ACC – accuracy

we can calculate the accuracy according to the following formula

$$ACC = \frac{TP+TN}{TP+FP+TN+FN}$$

We can also put the above information in the table, where the rows contain elements from the main class and subordinate class, respectively, while the columns contain elements classified by the classifier into the main class and subordinate class, respectively.

Elements from		classified by the classifier as main class	subordinate class
main class	P	TP	FN
subordinate class	N	FP	TN

Table 5.2 Table of parameters

Accuracy calculated for the test objects from the main class is called sensitivity (TPR – true positive rate), and the accuracy calculated for the test objects from a subordinate class we call specificity (TNR – true negative rate). In addition, we will consider FPR – false positive rate and PPV – precision. Using the above notion we can calculate the mentioned parameters according to the following formulas

$$TPR = \frac{TP}{P} = \frac{TP}{TP+FN}$$

$$TNR = \frac{TN}{N} = \frac{TN}{TN+FP}$$

$$FPR = \frac{FP}{N} = \frac{FP}{FP+TN}$$

$$FPR = 1 - TNR$$

$$PPV = \frac{TP}{TP+FP}$$

If the sensitivity is unsatisfactory, e.g., in medicine when trying to predict the occurrence of a disease of a patient, it may turn out that the sensitivity of the classification to the main class ”sick” is too low, we can balance between sensitivity and specificity, i.e., increasing sensitivity at the expense of decreasing specificity. This approach leads to the concept of the ROC curve (receiver operating characteristic curve), where each point of the ROC curve corresponds to one setting of the classifier's performance (the parameter t).

ROC shows the dependence of sensitivity on error of the first type (FPR) during calibration of the classifier (at various threshold settings).

For classifiers with this property (sensitivity and specificity regulation), the AUC parameter was used to assess their quality. AUC is the measure of the quality of a classifier which is the area under the ROC curve (cf. [94, 250]). The greater is the AUC value the better is the classifier.

Another measure of the quality of a classifier is F1-score. The F1-score combines the precision and recall of a classifier into a single metric by taking their harmonic mean. It is primarily used to compare the performance of two classifiers. Suppose that classifier A has a higher recall, and classifier B has higher precision. In this case, the F1-scores for both the classifiers can be used to determine which one produces better results.

The F1-score of a classification model is calculated as follows:

$$F1 = \frac{2(PPV * TPR)}{PPV + TPR}.$$

5.2 *k-NN* algorithm

In 1951 Fix and Hodges introduced a non-parametric method for pattern classification that has since become known as the *k*-nearest neighbor algorithm [95]. Next, some of the formal properties of the *k*-nearest neighbor rule were obtained [50]. The *k-NN* algorithm is a method for classifying objects based on the *k* closest training examples in a feature space. An object is classified by a majority vote of its neighbors, with the object being assigned to the class most common amongst its *k* nearest neighbors (*k* is a positive integer). If $k = 1$, then the object is simply assigned to the class of its nearest neighbor. It is a type of instance-based learning.

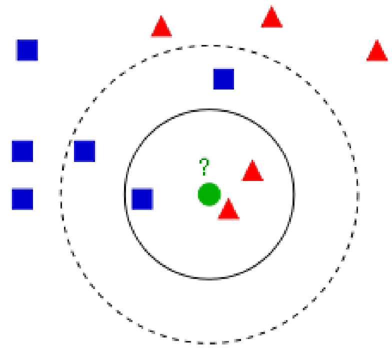

Fig. 5.2 k-NN classifier

For classification also a useful technique can be used, to assign weight to the contributions of the neighbors, so that the nearer neighbors contribute more to the decision than the more distant ones. For example, a common weighting scheme consists in giving each neighbor a weight of $1/d$, where d is the distance to the neighbor [91]. There were considered also another methods [28, 136, 139] and their applications [21].

Chapter 6
Aggregation of uncertainty from many classifiers by uninorms

It is clear that there is no classification of the Universe that is not arbitrary and full of conjectures. The reason for this is very simple: we do not know what kind of thing the universe is.

J.L. Borges

In this chapter we want to present the concept of uncertainty area of classifiers and an algorithm that uses uninorms to minimize the area of uncertainty in the prediction of new objects by complex classifiers (cf. [84]).

The classifier assigns to the object a certain weight (classification coefficient) to classify the object. For a set range of the threshold parameter $t \in (0,1)$, if the classification weight of the test object obtained from the classifier is greater than t, the object is classified into the main class (e.g., YES). However, if the weight is less than or equal to t, then the object is classified into a subordinate class (e.g., NO). However, for some neighborhood threshold t very small differences in the classification weight can lead to opposing decisions. In order to avoid the incorrect classification, we propose to introduce an uncertainty area, which if the classifier returns the classification weight from the certain neighborhood of a threshold, will lead to abstain from the decision.

In other words, in the case of classifying test objects, the so-called area of uncertainty is considered for which we abstain from the decision because we are not sure enough about it. Thanks to this, the classifier may make fewer mistakes while classifying, but from time to time, instead of the decision value, the classifier returns "I do not make decisions" or "I do not know".

Since the most errors of classification are made when the classification weight is close to the threshold parameter, we will refrain from the decision for this area. In this situation, the aforementioned ROC curve generation concept can be modified by introducing the uncertainty area. For this purpose, instead of simple threshold parameter t, we consider parameter ε such that $\varepsilon \in [0, \min(t, 1-t)]$. For the set value of parameters t and ε, classification of the test objects is performed in such a way that if the classification weight of the test object obtained from the classifier is greater than $t+\varepsilon$, then the object is classified into the main class (e.g. YES). On the other hand, if the classification weight is less than or equal to $t-\varepsilon$, then the object is classified into the subordinate class (e.g., NO).

In other cases, the object is classified into the so-called uncertainty area (see Figure 6.1). Similarly as before, for the calculation of the global classification quality of the given classifier with the parameters t and ε, we use classification accuracy

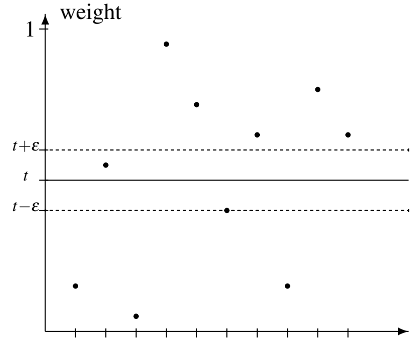

Fig. 6.1 The uncertainty area

for objects from the main class called sensitivity and accuracy calculated for the test objects from a subordinate class (specificity). In addition, for each experiment we obtain a third parameter, which we call a measure of the uncertainty of classification, which is a quotient of the number of test objects classified to the uncertainty area and the number of all test objects.

When classifying objects, we can construct different classifiers. Often the decisions obtained differ for some elements. Therefore, a conflict appears between the classifiers that operate on the basis of different sources or parameters, which must be resolved in order to finally classify the test object. For this purpose we suggest aggregation of values obtained by the individual classifiers using uninorms. As a result, we build a new compound classifier, which additionally reduces the measure of uncertainty area, and thus increases the coverage (i.e. the number of classified objects) of the entire test data.

6.1 Classifiers using different aggregations

When classifying objects, we can construct different classifiers (based on different systems or based on different data sources, e.g., using several diagnostic devices – see [19, 24, 50]). Often the decisions obtained differ for a certain class of test elements. Therefore, a conflict appears between the classifiers that operate on the basis of different sources or parameters, which must be resolved in order to finally classify the test object. To get a final decision, we should create a new classifier that will take into account previous results. One way to obtain complex classifiers is to use an aggregation of the values obtained by individual classifiers. As a result, we build a new compound classifier.

In this chapter, we suggest aggregating of the classification weights obtained by individual classifiers, and we propose two algorithms.

The first is when we use the means, denoted by the Algorithm M (Algorithm 1) and, in fact, we get a complex classifier.

Algorithm 1: Classification of a test object by the M classifier

---

Input:
1. training data set represented by decision table $\mathbf{T} = (U, A, d)$,
2. collection $C_1, \ldots, C_m$ of classifiers,
3. test object u,
4. aggregation M (mean),
5. threshold parameters t and ε.

Output: The membership of the object u to the "main class" or to the "subordinate class" or "no decision"

begin
    for $i := 1$ **to** m **do**
        Compute a certain weight ("main class" membership probability) for the given test object u using the classifier C_i and assign it to p_i
    end
    Determine the final weight p for the object u by aggregating (with a use of the mean M e.g., arithmetic mean) the weights $p_1, \ldots, p_m$.
    if $p > t + \varepsilon$ **then**
        return *u belongs to the "main class"*
    else
        if $p < t - \varepsilon$ **then**
            return *u belongs to the "subordinate class"*
        else
            return *we abstain from the decision*
        end
    end
end

Unfortunately, if we apply a quality assessment method that takes into account the uncertainty area, then it turns out that the measures of the uncertainty area for the M classifier is very high, which will be visible in the results of the experiments (see e.g. Table 6.4).

This is due to Lemma 1.1, because for example, for two classifiers with classification weights p_1 and p_2 that classify an object to the uncertainty area (weights p_1, p_2 belong to the interval $[t - \varepsilon, t + \varepsilon]$) our classifier M will classify the object to the uncertainty area (the weight p of the aggregated M classification will belong to the same interval). In addition, if only one of the weights will belong to the interval $[t - \varepsilon, t + \varepsilon]$, the weight of the final classification may belong to that interval.

Thus, by creating a classifier in this way, we increase the measure of the uncertainty area, not necessarily significantly increasing the accuracy of classification of the new classifier M in relation to the accuracy of aggregated classifiers.

Therefore, we propose also another method for aggregating classifiers based on uninorms with the neutral element being a value from the interval $(0, 1)$. This method will be denoted by Algorithm U (Algorithm 2).

Algorithm 2: Classification of a test object by the U classifier

Input:
1. training data set represented by decision table $\mathbf{T} = (U, A, d)$,
2. collection $C_1, ..., C_m$ of classifiers,
3. test object u,
4. uninorm U,
5. threshold parameters t and ε.

Output: The membership of the object u to the "main class" or to the "subordinate class" or "no decision"

begin
    for $i := 1$ **to** m **do**
        Compute a certain weight ("main class" membership probability) for the given test object u using the classifier C_i and assign it to p_i
    end
    Determine the final weight p for the object u by aggregating (with a use of the uninorm U e.g., representable uninorm) the weights $p_1, ..., p_m$.
    if $p > t + \varepsilon$ **then**
        return *u belongs to the "main class"*
    else
        if $p < t - \varepsilon$ **then**
            return *u belongs to the "subordinate class"*
        else
            return *we abstain from the decision*
        end
    end
end

We assume here that the neutral element e of a uninorm U will be equal to the threshold parameter t. Then, using Theorem 1.10, Lemma 1.4 and Example 1.9, for two classifiers with classification weights p_1 and p_2 belonging to the interval $[0, t]$ our classifier U assigns the classification weight of the object, which is less than or equal to the $\min(p_1, p_2)$. That is, if at least one of the weights $p_1,\ p_2$ is less than $t - \varepsilon$ then the object will be classified to the subordinate class. In other cases, the object can be classified to the subordinate class or to the uncertainty area.

If the weights p_1 and p_2 belong to the interval $[t, 1]$, then the classifier U assigns the classification weight of the object, which is greater than or equal to the $\max(p_1, p_2)$. It means that if at least one of the weights $p_1,\ p_2$ is greater than $t + \varepsilon$, then this object will be assigned to the main class. In other cases, the object can be classified to the main class or to the uncertainty area.

In both cases the degree of membership of the object to the main class or subordinate class is increased. If the object is classified to the uncertainty area with weights p_1 and p_2, such that $p_1 < t < p_2$, then the classifier U will classify the object to the uncertainty area.

6.2 Experiment

Now, we present detailed information about the experiments. The experiments have been performed on data sets obtained from UC Irvine (UCI) Machine Learning repository [252]. Table 6.1, shows the summary of the characteristics of the data sets. The considered attributes have numerical values only.

UCI data	Objects	Attributes	Classes
australian	690	15	2
biodeg	1055	43	2
breast_cancer	699	11	2
diabetes	768	9	2
german	1000	25	2
ozone	2536	74	2
parkinson	1040	29	2
red_wine	1599	12	2
rethinopathy	1151	20	2

Table 6.1 Experimental data set details

Here, we provide a brief description of the considered data sets.

- *Australian* (Australian Credit Approval) This file concerns credit card applications. All attribute names and values have been changed to meaningless symbols to protect confidentiality of the data. This dataset contains a good mix of attributes – continuous, nominal with small numbers of values, and nominal with larger numbers of values.
- *Biodeg* (QSAR Biodegradation Data Set, cf. [170]) – data set containing values for 41 attributes (molecular descriptors) used to classify 1,055 chemicals into 2 classes (ready and not ready biodegradable). The data have been used to develop QSAR (Quantitative Structure Activity Relationships) models for the study of the relationships between chemical structure and biodegradation of molecules.
- *Breast cancer* (Breast Wisconsin Original Data Set) – the creator is Dr. William H. Wolberg (physician, University of Wisconsin Hospitals Madison, Wisconsin, USA, [258]). The classification is into classes benign or malignant.
- *Diabetes* – Diabetes patient records were obtained from two sources: an automatic electronic recording device and paper records.
- *German* (German Credit Data) – classifies people described by a set of attributes as good or bad credit risks.
- *Ozone* (Ozone Level Detection Data Set) – two ground ozone level data sets are included in this collection. One is the eight hour peak set, the other is the one hour peak set. Those data were collected from 1998 to 2004 at the Houston, Galveston and Brazoria area. The following are specifications for several most important attributes that are highly valued by Texas Commission on Environmental Quality (TCEQ): O 3 – local ozone peak prediction, Upwind – upwind

ozone background level, EmFactor – precursor emissions related factor, Tmax – maximum temperature in degrees Fahrenheit, Tb – base temperature where net ozone production begins (50 Fahrenheit), SRd – solar radiation total for the day, WSa – wind speed near sunrise (using 09–12UTCforecastmode), WSp – wind speed mid-day (using 15–21 UTC forecast mode). More details can be found in the relevant paper [270].

- *Parkinson* (Parkinson Speech Dataset with Multiple Types of Sound Recordings Data Set) – the training data belongs to 20 Parkinson's Disease (PD) patients and 20 healthy ones. From all patients, multiple types of sound recordings are taken.
- *Red wine* (Wine Quality Data Set) – two data sets are included, related to red and white "vinho verde" wine samples, from the north of Portugal. The goal is to model wine quality based on physicochemical test.
- *Retinopathy* (Diabetic Retinopathy Debrecen Data Set) – this dataset contains features extracted from the Messidor image set to predict whether an image contains signs of diabetic retinopathy or not. All features represent either a detected lesion, a descriptive feature of a anatomical part or an image-level descriptor. The binary result of quality assessment is bad quality or sufficient quality. The underlying method image analysis and feature extraction as well as the classification technique is described in [9].
- *Spam* (Spambase Data Set) – the collection of spam e-mails coming from postmaster and individuals who had filed spam. The collection of non-spam e-mails coming from filed work and personal e-mails, and hence the word 'george' and the area code '650' are indicators of non-spam. These are useful when constructing a personalized spam filter. One would either have to blind such non-spam indicators or get a very wide collection of non-spam to generate a general purpose spam filter. The last column of Spam denotes whether the e-mail was considered spam (1) or not (0), i.e. unsolicited commercial e-mail. Most of the attributes indicate whether a particular word or character was frequently occurring in the e-mail. The run-length attributes (55—57) measure the length of sequences of consecutive capital letters. Creators are Mark Hopkins, Erik Reeber, George Forman, Jaap Suermondt from Hewlett-Packard Labs, 1501 Page Mill Rd., Palo Alto, CA 94304.

To get the C_i classifiers we use the k-nearest neighbor algorithm with different values of k. In this algorithm the Euclidean metric is applied for measuring distances. Each data set is divided into two training and test parts, in the proportion of 50% to 50%. The training part of the data is used to construct the C_i classifiers. Each experiment is repeated 20 times and the average AUC and standard deviation are reported using test part of data. We assume that all analyzed data have only two decision classes. Furthermore, for Algorithm M we use arithmetic mean and for Algorithm U we use a representable uninorm with the appropriate neutral element. The results represent the largest AUC for a given algorithm with a fixed uncertainty area. In this chapter, we will mainly present the results for the data from the sets *diabetes*, *biodeg*, *german* and *red wine*.

Consecutively, Tables 6.2–6.5 show examples of experimental results for *diabetes*, *red wine*, *biodeg* and *german* data using Algorithms 1–2.

Method	measures of the uncertainty area	STDDEV m.u.a.	AUC	STDDEV AUC
M	0.204	0.003	0.8	0.010
M	0.251	0.003	0.817	0.007
U (e=0.4)	0.134	0.004	0.789	0.009
U (e=0.5)	0.148	0.004	0.792	0.010

Table 6.2 Example of results of experiments for *diabetes* data set

Method	measures of the uncertainty area	STDDEV m.u.a.	AUC	STDDEV AUC
M	0.228	0.007	0.842	0.007
M	0.266	0.001	0.863	0.005
U (e=0.6)	0.154	0.007	0.821	0.006
U (e=0.4)	0.154	0.005	0.82	0.005

Table 6.3 Example of results of experiments for *red wine* data set

Method	measures of the uncertainty area	STDDEV m.u.a.	AUC	STDDEV AUC
M	0.143	0.003	0.92	0.005
M	0.169	0.005	0.929	0.005
U (e=0.4)	0.08	0.003	0.909	0.006
U (e=0.5)	0.097	0.003	0.911	0.004

Table 6.4 Example of results of experiments for *biodeg* data set

Method	measures of the uncertainty area	STDDEV m.u.a.	AUC	STDDEV AUC
M	0.203	0.003	0.722	0.012
M	0.275	0.002	0.774	0.01
U (e=0.5)	0.166	0.004	0.721	0.01
U (e=0.4)	0.15	0.004	0.717	0.013

Table 6.5 Example of results of experiments for *german* data set

As we can see in Tables 6.2–6.5 the Algorithm M yields a slightly larger AUC than Algorithm U. However, in each case, the measure of area of uncertainty using

Algorithm M is much larger than the measure of area of uncertainty when using Algorithm U. In fact, if we use the U Algorithm, we will get a more accurate classification.

The experiments were performed with a fixed width of the uncertainty area. As we can see in Tables 6.3 and 6.2, in the case of data from the *red wine* and *diabetes sets*, the amount of data that are in the uncertainty area is about 10–12 percentage points greater when using Algorithm 1 than in the case of using Algorithm 2 , while in the case of the *biodeg* and *german* data sets (see Table 6.4 and 6.5) it is a difference of 5–8 percentage points. It means that the use of Algorithm M almost doubles the number of unclassified objects in relation to Algorithm U. However, the difference in the AUC measure ranges from 0.001 (in the case of the *german* base) to 0.05 (in the case of the *red wine* base). Therefore, we believe that it is reasonable to use Algorithm U.

Chapter 7
Method of building multi-classifiers based on uninorms

Crude classifications and false generalizations are the curse of the organized life.

H.G. Wells

In this chapter the method for selecting classifiers to build a multi-classifier (an ensemble classifier) is presented. When classifying objects, we can construct different classifiers. Sometimes we get many classifiers that classify an object based on various premises (attributes) or different sources. Often the decisions obtained differ for some elements. Therefore a new classifier is being built that takes into account the weight of individual classifiers. Many of them are of low quality. Among other things, for this reason, in general, it gives better results than individual classifiers. The use of all classifiers (especially those of low quality) does not always give satisfactory results. However, the use of all classifiers and their later aggregation is sometimes very expensive. Therefore, we present a method that allows to eliminate some classifiers while increasing the quality of classification.

Our approach is characterized by, compared to the majority of existing ones, that classifiers are not only aggregated, but dynamically selected when testing a particular test object. The purpose of this selection is to improve the global quality of the classification and it is based on the raw results of the test object classification by all aggregated classifiers or based on certain selection parameters learned from training data. This approach is based on the paper [23].

Here we use the aggregation method based on the arithmetic mean and the representable uninorm for the experiments. The reason for this is that these aggregations in practice give good classification results obtained by multi-classifiers based on them, often better than other aggregations known from the literature.

7.1 Classification algorithm

When classifying objects, we can construct different classifiers (based on different systems or based on different data sources, e.g., using several diagnostic devices). Often the decisions obtained differ for a certain class of test elements. Therefore, a conflict appears between the classifiers that operate on the basis of different sources or parameters, which must be resolved in order to finally classify the test object. To get a final decision, we should create a new classifier that will take into account previous results. For this purpose we suggest aggregation of values obtained by the individual classifiers. As a result, we build a new compound classifier.

In this chapter, we use aggregating of the classification weights obtained by individual classifiers, and we propose Algorithm 3 (method WAS – weight arithmetic mean selection). This algorithm uses M aggregation, which is based on the arithmetic mean.

Algorithm 3: Classification of a test object by the M classifier

Input:
1. training data set represented by decision table $\mathbf{T} = (U, A, d)$,
2. collection $C_1, ..., C_m$ of classifiers,
3. test object u,
4. aggregation M,
5. threshold parameter t, e.g., $t = 0.6$.

Output: The membership of the object u to the "main class" or to the "subordinate class"

```
begin
    for i := 1 to m do
        Compute a certain weight ("main class" membership probability) for the given
          test object u using the classifier C_i and assign it to p_i
    end
    Determine the final weight p for the object u by aggregating (with a use of the
      aggregation operator M e.g., arithmetic mean) the weights p_1,...,p_m.
    if p > t then
        return u belongs to the "main class"
    else
        return u belongs to the "subordinate class"
    end
end
```

7.1.1 Modification of the algorithm

Some classifiers assigns a weight to an object that differ from the weights of other classifiers or their aggregation obtained in Algorithm 3.

That is why we decided to check their impact on the quality of the classification. In other words, we decided to check whether eliminating these classifiers would improve the quality of the classification.

The problem that appeared here was the choice of classifiers that we will use for classification, or in other words – the choice of classifiers that we will remove.

Here we propose four methods to choose from classifiers which we will use in the further part of the classification.

The first method is to choose those classifiers that give the weights closest to the value obtained in Algorithm 3 , i.e., those that are the most distant from the aggregate value are rejected (method WTS – weight threshold selection, see Algorithm 4).

Algorithm 4: Classification of a test object by the WTS classifier.

Input:

1. data set represented by decision table $\mathbf{T} = (U, A, d)$, with *card* $U = n$,
2. collection $C_1, ..., C_m$ of classifiers,
3. test object u,
4. aggregation M,
5. threshold parameter t, e.g., $t = 0.6$,
6. parameter ε, e.g., $\varepsilon = 0.8$.

Output: The membership of the object u to the ”main class” or to the ”subordinate class”

begin
- **for** $i := 1$ **to** m **do**
  - Compute a certain weight (”main class” membership probability) for the given test object u using the classifier C_i and assign it to p_i
- **end**
- Determine the weight p' for the object u by aggregating (with a use of the aggregation operator M e.g., arithmetic mean) the weights $p_1,...,p_m$.
- **for** $i := 1$ **to** m **do**
  - Compute a distance d_i between p' and p_i for the given test object u
- **end**
- Choose the classifiers for which $d_i < \varepsilon$. In this way we receive the sets $K = \{C_{s_1}, ..., C_{s_k}\}$
- Determine the final weight p for the object u by aggregating (with a use of the aggregation operator M) the weights $p_{s_1},...,p_{s_k}$.
- **if** $p > t$ **then**
  - **return** *u belongs to the ”main class”*
- **else**
  - **return** *u belongs to the ”subordinate class”*
- **end**

end

Unfortunately, for small values of ε we can get an empty set of K. That is why we suggest modifying this algorithm by selecting a certain percentage of classifiers for each test object. The selection method will be presented in Algorithm 5 (method WPS – weight percent selection).

The third construction, which differs from the previous two, consists in the selection of classifiers that have been recognized as the most stable based on the training set. Based on training data, we determine accuracy, sensitivity and specificity for each classifier (see Figure 7.1). We choose those for which the distance between the point of intersection of sensitivity and specificity and the maximum value of accuracy is the smallest. The selection method will be presented in Algorithm 6 (method SSAS – sensitivity, specificity, accuracy selection). Then, we aggregate the selected

Algorithm 5: Classification of a test object by the WPS classifier.

Input:
1. data set represented by decision table $\mathbf{T} = (U, A, d)$, with *card* $U = n$,
2. collection $C_1, ..., C_m$ of classifiers,
3. test object u,
4. aggregation M,
5. threshold parameter t, e.g., $t = 0.6$,
6. parameter $r \in (0, 1]$.

Output: The membership of the object u to the ”main class” or to the ”subordinate class”

begin
  for $i := 1$ **to** m **do**
    Compute a certain weight (”main class” membership probability) for the given test object u using the classifier C_i and assign it to p_i
  end
  Determine the weight p' for the object u by aggregating (with a use of the aggregation operator M e.g., arithmetic mean) the weights $p_1,...,p_m$.
  for $i := 1$ **to** m **do**
    Compute a distance d_i between p' and p_i for the given test object u
  end
  Choose $100 \cdot r$ % classifiers that are closest to the aggregate value p'. In this way we receive the sets $K = \{C_{s_1}, ..., C_{s_k}\}$
  Determine the final weight p for the object u by aggregating (with a use of the aggregation operator M) the weights $p_{s_1},...,p_{s_k}$.
  if $p > t$ **then**
    return *u belongs to the ”main class”*
  else
    return *u belongs to the ”subordinate class”*
  end
end

weights using the arithmetic mean. The weight obtained in this way is used to classify the object to the main class or subordinate class, depending on the assumed threshold parameter.

The fourth construction is very similar to the third. As before, we select the classifiers for which the distance between the point of intersection of sensitivity and specificity and the maximum value of accuracy is the smallest. The selection method will be presented in Algorithm 7 (method SSASU – sensitivity, specificity, accuracy selection using a uninorm).

Then, we aggregate the selected weights using a uninorm (in our consideration we use a representable uninorm). The weight obtained in this way is used to classify the object to the main class or subordinate class, depending on the assumed threshold parameter.

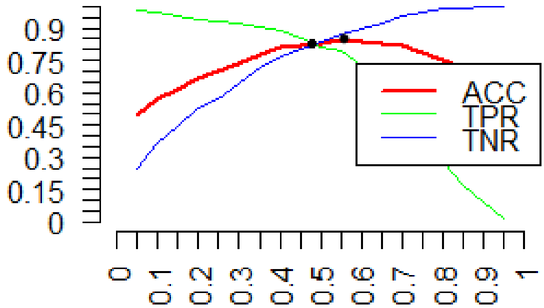

Fig. 7.1 Graph of dependence of accuracy, sensitivity and specificity on the threshold parameter s

7.2 Experiments

According to the above algorithms, we have m classifiers as input. Because of this, we need a set of classifiers. Here we will use *k-NN* classifiers.

The experiments have been performed on data sets obtained from UC Irvine (UCI) Machine Learning repository. They are listed in Table 6.1 (see also Section 6.2.

To get the C_i classifiers we use the k-nearest neighbor algorithm with different values of k. In this algorithm the Euclidean metric is applied for measuring distances. Each data set is divided into two training and test parts, in the proportion of 50% to 50%. The training part of the data is used to construct the C_i classifiers. Each experiment is repeated 10 times and the average AUC and standard deviation are reported using the test part of data. We assume that all analyzed data have only two decision classes. In this work, we will mainly present the results for the data from the sets *diabetes* and *red wine*.

Consecutively, Table 7.1 shows examples of experimental results for *diabetes* data using Algorithms 3–7. Table 7.2 and 7.3 show the average AUC for the results of experiments for individual algorithms for the *diabetes* and *red wine* data sets, and Figure 7.2, 7.3 show the graphical interpretation of these results.

As we can see in Figure 7.2 and Figure 7.3 the methods SSAS and SSASU gives the highest of all the AUC and is the most stable value without deviation up or down. It means that this two (SSAS and SSASU) are the most effective, and that there are stable values obtained using Algorithm SSAS and SSASU, while the other methods have high AUC dispersion. In addition, we see that the worst results are obtained for the WTS Algorithm. Therefore, we presented a method that allows to eliminate some classifiers while increasing the quality of classification. In addition, if the classifiers use different attributes, using the reduction of classifiers we can decide which attributes are important and which can be omitted.

Algorithm 6: Classification of a test object by the SSAS classifier.

Input:
1. training data set represented by decision table $\mathbf{T} = (U, A, d)$, with *card* $U = n$,
2. collection $C_1, \dots, C_m$ of classifiers,
3. collection of weight thresholds $T = \{t_1, \dots, t_z\}$ used during computation of ROC curve,
4. test object u,
5. aggregation A – arithmetic mean,
6. threshold parameter t, e.g., $t = 0.6$,
7. parameter ε, e.g., $\varepsilon = 0.8$.

Output: The membership of the object u to the "main class" or to the "subordinate class"

begin
  for $i := 1$ **to** m **do**
    For the C_i classifier, perform calculations such as for determining the points of the ROC curve based on the training table and the collection T, obtaining a list of points $LP_i = P_1, \dots, P_z$; assume that each point on this list is the triplet (sensitivity, specificity, accuracy).
    Based on the list of LP_i determine the point $P_{ss} = \{sen_{ss}, spec_{ss}, acc_{ss}\} \in LP_i$ for which the distance between sensitivity and specificity is the smallest (closest to the point of intersection of sensitivity and specificity)
    Based on the list of LP_i determine the point $P_a = \{sen_a, spec_a, acc_a\} \in LP_i$ for which the value of accuracy is the highest
    Compute the distance d_i between $\frac{sen_{ss}+spec_{ss}}{2}$ and acc_a.
  end
  Choose $\varepsilon \cdot m$ classifiers for which the distance d_i is the smallest. In this way we receive the sets $K = \{C_{s_1}, \dots, C_{s_k}\}$
  for $i := 1$ **to** k **do**
    Compute a certain weight ("main class" membership probability) for the given test object u using the classifier C_{s_i} and assign it to p_{s_i}
  end
  Determine the final weight p for the object u by aggregating (with a use of the arithmetic mean A) the weights $p_{s_1}, \dots, p_{s_k}$.
  if $p > t$ **then**
    return *u belongs to the "main class"*
  else
    return *u belongs to the "subordinate class"*
  end
end

Algorithm 7: Classification of a test object by the SSASU classifier.

Input:
1. training data set represented by decision table $\mathbf{T} = (U, A, d)$, with *card* $U = n$,
2. collection $C_1, \ldots, C_m$ of classifiers,
3. collection of weight thresholds $T = \{t_1, \ldots, t_z\}$ used during computation of ROC curve,
4. test object u,
5. uninorm U,
6. threshold parameter t, e.g., $t = 0.6$,
7. parameter ε, e.g., $\varepsilon = 0.8$.

Output: The membership of the object u to the "main class" or to the "subordinate class"

begin
  for $i := 1$ **to** m **do**
    For the C_i classifier, perform calculations such as for determining the points of the ROC curve based on the training table and the collection T, obtaining a list of points $LP_i = P_1, \ldots, P_z$; assume that each point on this list is the triplet (sensitivity, specificity, accuracy).
    Based on the list of LP_i determine the point $P_{ss} = \{sen_{ss}, spec_{ss}, acc_{ss}\} \in LP_i$ for which the distance between sensitivity and specificity is the smallest (closest to the point of intersection of sensitivity and specificity)
    Based on the list of LP_i determine the point $P_a = \{sen_a, spec_a, acc_a\} \in LP_i$ for which the value of accuracy is the highest
    Compute the distance d_i between $\frac{sen_{ss}+spec_{ss}}{2}$ and acc_a.
  end
  Choose $\varepsilon \cdot m$ classifiers for which the distance d_i is the smallest. In this way we receive the sets $K = \{C_{s_1}, \ldots, C_{s_k}\}$
  for $i := 1$ **to** k **do**
    Compute a certain weight ("main class" membership probability) for the given test object u using the classifier C_{s_i} and assign it to p_{s_i}
  end
  Determine the final weight p for the object u by aggregating (with a use of the uninorm U) the weights $p_{s_1}, \ldots, p_{s_k}$.
  if $p > t$ **then**
    return *u belongs to the "main class"*
  else
    return *u belongs to the "subordinate class"*
  end
end

Method	data reduction	AUC	STDDEV
WPS	0.5	0.813	0.002
SSAS	1.0	0.813	0.003
WTS	0.9	0.812	0.003
SSASU	1.0	0.812	0.004
WTS	1.0	0.812	0.003
WPS	0.2	0.811	0.003
WPS	0.7	0.811	0.003
SSAS	0.8	0.811	0.004
WPS	0.8	0.81	0.002
SSAS	0.3	0.81	0.006
SSASU	0.3	0.81	0.005
SSASU	0.6	0.81	0.003
SSAS	0.6	0.81	0.003
SSAS	0.7	0.81	0.004
WTS	0.8	0.809	0.003
SSAS	0.4	0.809	0.004
SSASU	0.5	0.809	0.004
SSAS	0.9	0.809	0.002
SSAS	0.2	0.808	0.004
WPS	0.6	0.807	0.004

Table 7.1 Example of results of experiments for *diabetes* data set

Method	average of AUC	STDDEV
WAS	0,765	0,071
WTS	0,752	0,065
WPS	0,762	0,076
SSAS	0,778	0,027
SSAS	0,777	0,028

Table 7.2 The average AUC for individual algorithms using data set *diabetes*

Method	average of AUC	STDDEV
WAS	0,781	0,081
WPS	0,768	0,062
WTS	0,782	0,078
SSAS	0,800	0,017
SSASU	0,800	0,016

Table 7.3 The average AUC for individual algorithms using data set *red wine*

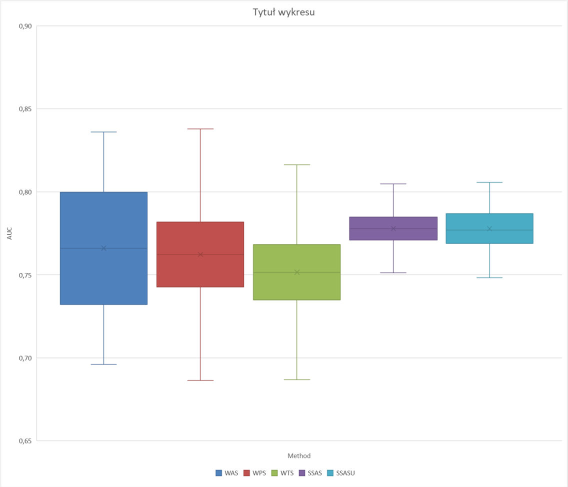

Fig. 7.2 The results of AUC for *diabetes*

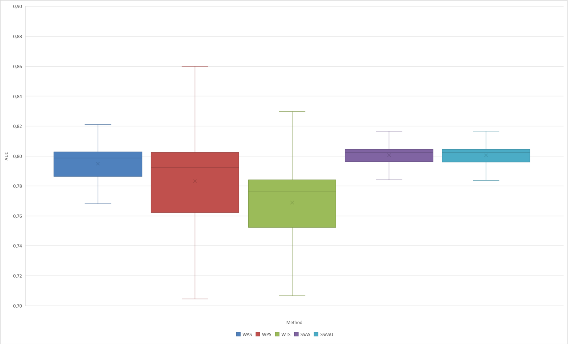

Fig. 7.3 The results of AUC for *red wine*

Chapter 8
Use of uninorms to classify phishing emails

Finding an alternative to supplement military ways of resolving international conflicts has been taken up by many people skilled in various areas such as political science, economics, social studies, modelling and simulation, intelligence and expert systems, military strategy and weaponry as well as private business and industry.

H. Chestnut

In this chapter, we will create models to detect phishing e-mails. Prediction of phishing e-mails can be based on two sources of information: e-mail content and links in e-mails. Although the links appear in the body of the e-mail, both of these sources of information are usually treated separately and require the development of special methods of their analysis. So, the analysis of datasets containing e-mails content and datasets containing links from e-mails results in two classifiers. The first one is able to classify the content of an e-mail by predicting phishing content. The second one allows you to predict whether any link in the message is a phishing link. If, when classifying a specific e-mail, both of the above classifiers match, the situation is clear. However, if they do not match, then we need to propose a method for resolving the conflict between these classifiers. This method can be based on various approaches. In this chapter, we propose a method based on the aggregation of the classification weights of both classifiers, using the methods of the fuzzy set theory. Both of the above-mentioned classifiers, together with the established method of resolving conflicts between them, can be treated as a complex classifier for recognizing phishing. Therefore, the use of an optimally selected method of resolving the above-mentioned conflict is of key importance for the effectiveness of the complex classifier.

In addition, it should be mentioned that company FreshMail does not have its own examples of phishing. Therefore, the prediction only applies to a fixed numerical level of suspicion that the e-mail is phishing. This is calculated based on the similarity to external phishing e-mails and external links from phishing e-mails.

8.1 The phishing problem

The Oxford Dictionary has defined phishing as “the fraudulent practice of sending e-mails purporting to be from reputable companies in order to induce individuals to reveal personal information, such as passwords and credit card numbers.”

One of the main ways of phishing is to send e-mails to potential victims. This problem is becoming more and more common.

In the literature, we can find many works dealing with the recognition of phishing e-mails (see e.g. [259] for a literature review). The proposed methods are to protect us from the effects of phishing by filtering e-mails or blocking links.

FreshMail (FM) is a Polish company specializing in comprehensive support for e-mail marketing activities, as well as providing a popular e-mail marketing tool. Every month, with its help, thousands of marketers from all over the world send over a billion messages.

When implementing the SendGuard project, FreshMail wants to be able to recommend, based on an intelligent analysis of the enormous data sets at the stage of creating an e-mail project by the marketer, whether the e-mail should be addressed to all recipients. In addition, in order to protect recipients, as part of the SendGuard project, FreshMail undertook the task of developing methods of predicting phishing e-mails in order to automatically filter them out before sending the e-mail (cf. [22]).

In our work, closely related to paper [22], we assume that the prediction of phishing e-mails can be based on two sources of information: e-mail content and links in e-mails. Although the links appear in the body of the e-mail, both of these sources of information are usually treated separately and require the development of special methods of their analysis. So, the result of e-mail content analysis and link analysis are two classifiers. The first one is able to classify the content of an e-mail by predicting phishing content. The second one allows you to predict whether any link in the message is a phishing link.

If, when classifying a particular e-mail, both of the above classifiers are consistent (both classify it as ”phishing” or both classify it as ”normal”), then the situation is clear. However, if they do not match (e.g. one classifies as ”phishing” and the other classifies as ”normal”), then we need to propose a method for resolving the conflict between these classifiers. This method can be based on various approaches. In this chapter, we propose a method based on the aggregation of the classification weights of both classifiers, using the methods of the fuzzy set theory.

Both of the above-mentioned classifiers, together with the established method of resolving conflicts between them, can be treated as a complex classifier for recognizing phishing. Therefore, the use of an optimally selected method of resolving the above-mentioned conflict is of key importance for the effectiveness of the complex classifier.

8.2 The methodology

The inspiration for proposing this methodology was the fact that FM does not have its own examples of phishing e-mails (sent by FM clients), but only examples of texts of such e-mails obtained from external sources. Therefore, it was assumed that a given e-mail is not obligatorily classified as ”phishing” or ”normal”, but it is examined to what extent the e-mail is similar to the phishing e-mails obtained from the outside. The interpretation of this similarity may be presented to analysts, who can make decisions related to stopping the sending of the suspicious e-mail. In order to determine the degree of similarity to phishing e-mails, the system assigns two weights to the degree of certainty that a given e-mail is phishing e-mail (*wt* – based on the analysis of the e-mail text, whereas *wl* – based on the analysis of the links in the e-mail). Each of these weights is a number in the range [0.1]. These weights are used to estimate the degree of certainty that a given e-mail is phishing e-mail. In order to easily illustrate and interpret the confidence level using the *wt* and *wl* weights, a number of concepts are defined to represent different levels of confidence in phishing recognition. For the purposes of this chapter, we will refer to these concepts as predictive concepts for phishing. Multiple phishing predictive concepts can be defined based on different *wt* and *wl* thresholds and different ways to combine them. They can have different marketing meanings. For example, one example of a predictive concept might be a concept defined by the following formula: $wt > 0.2$ and $wl > 0.9$. This concept includes all e-mails for which $wt > 0.2$ and $wl > 0.9$.

8.2.1 The method of calculating the weight for an e-mail based on the analysis of the e-mail text

One way to predict a specific e-mail as a phishing e-mail is to analyze the content of the e-mail. The analysis of the content of the e-mail should provide a weight that indicates the degree of similarity of the e-mail to phishing e-mails.

One of such methods may be the analysis of the presence in the e-mail of words characteristic for phishing e-mails, which were previously determined by a different method, i.e. based on the analysis of the difference in the frequency of occurrence of words in non-phishing e-mails and e-mails with phishing. A set of such words has been denoted in this paper by PHIS_WORDS. Examples of words in PHIS_WORDS are: update, improve, verify, block, detect and many others. Thus, the method of determining the weighting of an e-mail from a phishing point of view discussed in this section begins by analyzing the presence of specific phishing words in an e-mail, taking into account their order as well. As a result of the analysis, the result is a sequence of words characteristic of phishing, in which the order of the words corresponds to the order in which the words appear in the e-mail. Then, the determined sequence of words is matched to the analogous sequences previously determined for all available sample phishing e-mails. The method described in this paper uses the Needleman-Wunsch algorithm to match a fixed e-mail to a sample phishing e-mail (for more details on the Needleman-Wunsch algorithm you can see at [215]). More precisely, if we have a given sequence from the *m* test e-mail (*sm*), which we match to the sequence from a certain *pm* – phishing e-mail (*spm*), then by $NW(sm, spm)$ we denote the number which is the length of the *sm* sequence matching to *spm* using the Needleman-Wunsch algorithm. It is a natural number less than or equal to the length of the sequences *sm* and *spm*. By matching a sequence from a specific test e-mail m with different sequences from phishing e-mails (included in the PHIS collection), we get different match values, which strongly depend on the length of the phishing e-mail sequence. Therefore, to find a good (fairly complete) match to a phishing e-mail, we normalize the number of matched words by the length of the word sequence from the phishing e-mail. Thus, for a test e-mail m with a word sequence sm and a phishing e-mail pm with a word sequence spm, the value of the $MM(m,pm)$ match of the e-mail m to pm is expressed by the formula:

$$MM(m,pm) = NW(sm,spm)/length(spm)$$

We use the above measure to find the most suitable phishing e-mail for a given test e-mail m. If the m test e-mail and the phishing e-mail collection PHIS $= \{pm_1,...,pm_k\}$ are given, then the $MM(m,\text{PHIS})$ match of the e-mail m to the PHIS collection is calculated according to the formula:

$$MM(m,\text{PHIS}) = \max(MM(m,pm_i),\ for\ i = 1,...,k).$$

So, we obtain the value of the first classifier.

8.2.2 The method of calculating the weight for an e-mail based on the analysis of links in the e-mail

In addition to the analysis of the texts of the e-mails, the links contained in the e-mails were used to recognize phishing. Therefore, during the implementation of the project, a collection of LINKS5 links was used, prepared at an earlier stage of the project. It contains all links appearing in FM datasets as well as all links observed in the world in phishing e-mails, included during the period for which the experiments were performed. Links from phishing e-mails were obtained from various organizations collecting such links. In addition to the links themselves, a set of link properties has been added to the LINKS5 set for each link. Due to the fact that in the LINKS5 file there is also an attribute with the information about the time of the first observation of the link in e-mails, the LINKS5 file can be divided into sets using cut-off dates.

In addition, based on the LINKS5 set, we can create decision tables that will generate the CDLINKS5 classifier used to predict phishing for individual links in the e-mail. Because there are often many links in e-mails, and in order to determine whether a phishing link is inserted in a given e-mail, you need to make predictions on all links from the e-mail. That is why we define a new type of classifier for prediction of links in e-mails, which we mark with MCDLINKS5. It works in such a way that it first predicts the weights of belonging to the ”bad” class of all links from the tested e-mail. Then it determines the prediction result for the tested e-mail. In other words, the MCDLINKS5 classifier for a given e-mail returns the weight of the most suspicious link from that e-mail.

8.3 Algorithm for constructing classifiers for identifying phishing

In this section, we describe the classification algorithm (classifier) for e-mail prediction and the proposed procedure for testing the introduced classifier. It is streaming, i.e. the classifier is trained before each testing day *d* based on the 30 days preceding that day. Using this approach, to test over 40 million transactional e-mails (this is how many e-mails are counted by the analyzed data), you need to stream classifiers for many days. In the experiment, the results of which we present in this chapter, the days from February 1, 2021 to November 30, 2021 (303 days) were analyzed. From this period, there were over 40 million transactional e-mails.

Algorithm 8: Classification of e-mails by the classifier

Input:

1. the predictive concept of phishing C ,
2. DAYS – a collection of all days
3. PHIS – a sample of all available phishing e-mails,
4. PHIS_TRAIN – a training sample of phishing e-mails
5. PHIS_WORDS – a "phishing words" in e-mails,
6. LINKS5 – an array of phishing links
7. DSMT – table of transactional e-mails

Output: The membership of the e-mail m ∈DMST to the "phishing class"

1. ACTUAL_LIST = []
2. PREDICTED_LIST = []
3. **For** each day *d* from the DAYS collection

   a Generate DLINKS5_TR tables.
   b Generate DLINKS5_TRD tables.
   c Generate the MCDLINKS5_TR classifier.
   d Generate the MCDLINKS5_TTD classifier.
   e Generate a dataset for DSMT (d).
   f **For** each e-mail from DSMT (d)
   $wt1$ = NMM (m, PHIS_TR)
   $wt2$ = NMM (m, PHIS_TRD)
   $wl1$ = MCDLINK5_TR (m)
   $wl2$ = MCDLINK5_TRD (m)
   if RESOLVE $(wt2, wl2)$ = "phishing" **then** add the "phishing" decision to the ACTUAL_LIST
   else: add the decision "normal" to the ACTUAL_LIST
   if RESOLVE $(wt1, wl1)$ = "phishing" **then** add the "phishing" decision to the PREDICTED_LIST
   else: add the decision "normal" to the PREDICTED_LIST

4. Verify the correspondence of the ACTUAL_LIST and PREDICTED_LIST lists by calculating the necessary measures of the classifier's quality.

8.3.1 Explanation of notions

In this part we explain some of the notions used in Algorithm 8.

- The concept of predictive phishing C is defined for the wt and wl thresholds.
- DAYS denotes a collection of all days arranged chronologically from February 1, 2021 to November 30, 2021.
- PHIS denotes a collection of all available phishing e-mails.
- PHIS_TRAIN denotes a training set of phishing e-mails obtained by randomly separating 50% of lines from PHIS.
- PHIS_WORDS denotes a set of characteristic words appearing in the texts of phishing e-mails.
- LINKS5 denotes an array of links and their features, including links that appeared in 2021.
- DSMT denotes the table of transactional e-mails sent from FM and their attributes for 2021.
- ACTUAL_LIST denotes the decision list (”phishing” or ”normal”) describing the actual membership of transactional e-mails to the concept of C (at the beginning of the algorithm it is an empty list).
- PREDICTED_LIST denotes the decision list (”phishing” or ”normal”) describing the prediction of whether transactional e-mails belong to the concept C (at the beginning of the algorithm it is an empty list).
- The DLINKS5_TR tables is generated from the 30 days preceding day d.
- The DLINKS5_TRD tables is generated from the 30 days preceding day d and day d.
- The MCDLINKS5_TR classifier is generated for links based on DLINK5_TR, optimizing its creation parameters on the table DLINKS5_TR.
- The MCDLINKS5_TTD classifier is generated for links based on LINK5_TRD, optimizing its creation parameters on the table DLINKS5_TRD.
- The dataset DSMT (d) denotes the transactional mails from day d and it is generated from DSMT based on attributes.

- The $wt1$ weight is the weight generated for the m e-mail using the NMM (m, PHIS_TR) function including the PHIS_TR sample and words from the PHIS_WORDS collection.

- The $wt2$ weight is the weight generated for the m e-mail using the NMM (m, PHIS_TRD) function including the PHIS_TRD sample and words from the PHIS_WORDS collection.

- The $wl1$ weight is the weight generated for the m e-mail using the MCD LINK5_TR (m) function, for the most suspicious link from m by the classifier built for the DLINK5_TR table.

- The $wl2$ weight is the weight generated for the m e-mail using the MCD LINK5_TRD (m) function, for the most suspicious link from m by the classifier built for the DLINK5_TRD table.

8.3.2 Comparing the results of the classification

To calculate the prediction value for individual e-mails, it is enough to perform only part of the algorithm marked in black. However, here we will also want to evaluate the quality of the classifier. Therefore, we perform the entire algorithm, and the weights $wt1$ and $wl1$ will be used to test the quality of the classifier obtained.

8.3.3 Procedure RESOLVE

One of the most important elements of the algorithm is the RESOLVE procedure. This procedure has two weights as input, the weight wt obtained from the classifier based on the analysis of the e-mail text and the weight wl obtained from the classifier based on the analysis of links contained in the e-mail. On the basis of these weights, it decides to which class the tested e-mail belongs, i.e. to PHISHING or NORMAL. In this paper, to construct this procedure, we used weight aggregation methods that we recalled earlier in Section 1.2, such as arithmetic and geometric means, or triangular norms, nullnorms and uninorms from various classes. Since all of the above aggregations are commutative, we expect both classifiers to have the same decision threshold. However, if it were not so, we will scale one of the weights (for the purposes of this work, we assume that the information obtained from both classifiers is equally important). Then using the fixed aggregation for the weights wt and wl we get some value. Depending on this value, RESOLVE returns PHISHING or NORMAL.

8.4 Results

During the experiments in the RESOLVE procedure, we used the various aggregations described in Chapter 1. As we can see in Table 8.4, the best results were obtained for two aggregations: $U_{\min}$ uninorm and T_M t-norm.

Aggregation	ACC	PPV	TPR	F1 score
$U_{\min}$	0.99999	0.99975	1.00000	0.99987
$U_{\max}$	0.98942	0.99398	0.99161	0.99279
T_L	0.99999	0.97025	0.99558	0.98275
T_M	0.99999	0.99999	0.99999	0.99999
A_M	0.73264	0.98362	0.98362	0.77831

Table 8.1 The results for different aggregations

The proposed method of recognizing phishing mails differs from that proposed in the literature. Other authors usually had e-mails labeled phishing or normal in their works. This cannot be the case in this paper, because FM does not have its own examples of phishing. Therefore, the prediction only applies to a fixed numerical level of suspicion that the e-mail is phishing. This is done based on the similarity to external phishing e-mails and external links from phishing e-mails. This level can be set in various ways, depending on the current needs of analysts. It is worth adding that if the thresholds level is set high (e.g. 0.9), the number of suspicious e-mails will be lower than if the thresholds level is set lower (e.g. 0.6). In this chapter the main purpose of prediction is to select e-mails to be checked by experts in order to decide whether the e-mail should be sent or not. In the case of transactional e-mails, we have to be sure that the blocked e-mail is bad (phishing e-mail). Therefore, the above dependencies are useful here and the obtained results are satisfactory.

Chapter 9
Concluding remarks

The monograph aimed to present a full picture of uninorm theory relevant to the 2023 deadline, taking into account the latest developments. While finishing the book, I received an alert about new results that had just appeared. (...) And at this point I would like to apologize to all scientists whose results regarding uninorms were not included in this book, while being happy that the theory I deal with in my scientific work enjoys so much interest among scientists from all over the world.

To summarize this monograph, it should be pointed out that the reader will find here both full information about the structure of uninorms from known classes (which were compared at the end of the first chapter) as well as several applications of uninorms in the construction of classifiers.

Mathematically, uninorms are ordered semigroups on $[0,1]$ with a neutral element and/or solution of the associativity equation. The first solutions to this problem can be found in Abel (1826) where differentiable, symmetric, strictly increasing solutions in $\mathbb{R}$ were obtained. The next stages are Hölder's paper (1901) where Archimedean, naturally ordered semigroups were considered. In 1946 Climescu considered the sum of semigroups. Aczél in 1948 described continuous, cancellative semigroups for open and half open interval. Clifford in 1954 introduced the notion of ordinal sum of semigroups. Schweizer, Sklar in 1960 introduced the notion of triangular norms and conorms. Ling in 1965 described continuous, Archimedean semigroups for closed intervals and many others descriptions can be found in the literature.

Later in the literature we can find many considerations regarding this type of operations. Those that are close to uninorms include the papers [70], [141] and [52], where some kind of aggregation operators combining minimum and maximum were considered, and finally appeared as uninorms in the papers [107] and [268].

The idea of uninorms was deeper studied in [102], where the structure of such operators was analysed and two first classes of uninorms were introduced: uninorms in $\mathscr{U}_{\min}$ (given by minimum on $A(e)$), and $\mathscr{U}_{\max}$ (given by maximum on $A(e)$) – see Sec.1.7, and representable uninorms $\mathscr{U}_{rep}$ (extremely related with Dombi's operators introduced in [70], see also [71, 141]) – see Sec.1.9. After this, some other classes of uninorms were introduced and characterized. Some of theses classes are:

- Idempotent uninorms $\mathscr{U}_{id}$, those that satisfy $U(x,x) = x$ for all $x \in [0,1]$. In this family, $T = \min$ and $S = \max$. From results stated by Czogała and Drewniak [52], they were firstly studied by De Baets [57]. The characterization of idempotent uninorms is due to Martín et al. [177] and corrected by Ruiz-Aguilera et al. [235] – see Sec. 1.11.
- Representable uninorms, those that have an additive generator. They are the only uninorms that are continuous in $[0,1]^2 \setminus \{(0,1),(1,0)\}$. They were introduced by Dombi [70] and characterized by Fodor et al. [102], by Ruiz-Aguilera and Torrens [229], and by Fodor and De Baets [98] – see Sec. 1.9.
- Uninorms continuous in $(0,1)^2$, $\mathscr{U}_{cos}$, characterized by Hu and Li [124] and Drygaś [78] – see Sec.1.10.
- Uninorms with the continuous underlying operators $\mathscr{U}_{cuo}$. First presented in the 2003 Linz Seminar [100], where the general idea of the structure of this family of uninorms was presented (see also [61]).
- Uninorms with the continuous underlying operators, which have been characterized for the cases whenever T and S are strict or nilpotent [98, 154, 150].
- Uninorms locally internal on $A(e)$, that is, those uninorms that satisfy that $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$. They have been studied by Drewniak and Drygaś [72, 76, 79, 82] – see Sec. 1.12.
- Uninorms locally internal on $A(e)$, with the continuous underlying operators, characterized by Drygaś, Ruiz-Aguilera and Torrens [90] – see Sec. 1.13.7.
- Uninorms with the continuous underlying operators, described by Mesiarová-Zemánková using set-valued function [203], using a set of discontinuity points [202], using ordinal sum construction [200] and use separating functions – see Sec. 1.13.8.
- Uninorms locally internal on the boundary, for which some properties are given by [151] – see Sec. 1.14.
- Uninorms not locally internal on the boundary, which are presented in the papers [121, 260] – see Sec.1.15.
- Uninorms constructed by paving, whose constructions can be found in the papers [31, 284].

All the above classes were recalled in the first chapter, and their characterization was given. In addition, a description of uninorms with continuous underlying operations using a separating function was presented, and some properties of uninorms locally internal on the boundary and not locally internal on the boundary are presented. Especially the description using the separation functions seems to be the most intuitive and easy to describe.

Since uninorms on unit interval are always conjunctive or disjunctive ($U(0,1) = 0$ or $U(0,1) = 1$) they have been extensively studied in the framework of logical connectives and they have been used in fuzzy set theory and fuzzy logic [145]. In this line, uninorms are useful not only as conjunction and disjunction, but also in other tasks of fuzzy sets and fuzzy logic. For instance, they have been used to generate fuzzy implication functions [17], such as RU-implications [4, 59], (U,N)-implications [18] and QL or D-implications [186]. Uninorms have been also the basis for constructing new fuzzy logics as generalization of substructural logics

[209, 105, 210, 171]. These uninorms logics are obtained by generalizing the t-norm based approach to interpret conjunction and implication connectives by conjunctive uninorms and their residual or RU-implications.

In Chapter 4 we have presented ways to construct inverted fuzzy implications. As shown in the papers [248, 249, 246], among the typical examples of fuzzy implications, there is the problem of inversibility over the entire unit square. Since the implication family forms a lattice, one way to solve the problem is to find the largest or smallest implication with respect to the inverse implication in a given subset (the partition depends on the selected family of implications) and combine them to obtain the optimal implication. However, this is a very complex process. This is why we have shown that the (UN)-implication is invertible with respect to the antecedent as well as the conclusion, assuming that the uninorm is representable, and that the family of (UN)-implications form an ordered family with respect to the parameter e, with a fixed generator of a uninorm. Thus, we can more easily choose the appropriate implication for the problem under consideration.

In our further considerations (not included in this book), we plan to use the constructed operations in real-world problems. For this purpose, the PNeS system (see [247]) has been supplemented with appropriate modules and Petri Nets are constructed using uninorms and inverted fuzzy implications. Backward and forward reasoning simulations are also conducted.

Another application of uninorms is to use them in expert systems [33, 60] (where uninorms were used in the medical system MYCIN or the geological system PROSPECTOR), neural network, neural networks [148, 149, 163, 220, 221, 118], fuzzy systems modelling [265, 267], image processing [56, 112, 108], image compression [25], edge detection [109, 111, 110], decision making [266, 166], consensus [128, 157, 277], fuzzy integrals [142, 144] and other [140, 255].

In Chapters 6–7 we presented an application to creating multiclassifiers using uninorms. Two approaches are presented here. The first one allows for the selection of classifiers, so that only the weights of the best classifiers from our point of view are allowed into the aggregation process. In the second approach, we used the so-called the uncertainty area of the classifier, which allows us to not classify objects whose classifier weights are close to the threshold parameter (for such objects, the most classification errors are made during classification). As it turns out, the use of a uninorm to aggregate the weights of classifiers reduces the number of unclassified objects without reducing the quality of classification. All considerations were carried out for two-class classifiers obtained using the k-NN method. In the future, we plan to examine the effectiveness of these methods for other types of classifiers (larger number of classes and classifiers obtained using other methods). Finally, in Chapter 8, we presented the use of uninorms to classify phishing emails. The methods used allowed for increasing the effectiveness of dealing with the phishing problem.

If we use uninorm as logical connectives (conjunction or disjunction), we can check what logical properties they satisfy. This leads to different kinds of functional equations, like distributivity [180, 181, 224, 232, 74, 88, 134, 261], modularity [179], migrativity [182, 183, 242, 276, 158, 279, 280] and others [34, 67, 120, 225, 187, 152, 3, 205].

Due to the generalization of fuzzy sets, there was also a description of uninorms on interval valued fuzzy set [68], Atanassov's intuitionistic fuzzy set [69], chains [192, 234, 65, 240], lattices [137, 30, 46] and others [227, 251].

Finally, it should be pointed out that only some of the papers on uninorms (over two thousand papers according to Google Scholar) are mentioned in this book. More precisely, those for the unit interval contained the final results, and the others are those representative of the problem under consideration regarding uninorms.

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