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POLYNOMIAL ULTRADISTRIBUTIONS: DIFFERENTIATION AND LAPLACE TRANSFORMATION

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Abstract. We consider the multiplicative algebra $\mathsf{P}(\mathcal{G}'_+)$ of continuous scalar polynomials on the space $\mathcal{G}'_+$ of Roumieu ultradistributions on $[0,\infty)$ as well as its strong dual $\mathsf{P}'(\mathcal{G}'_+)$. The algebra $\mathsf{P}(\mathcal{G}'_+)$ is densely embedded into $\mathsf{P}'(\mathcal{G}'_+)$ and the operation of multiplication possesses a unique extension to $\mathsf{P}'(\mathcal{G}'_+)$, that is, $\mathsf{P}'(\mathcal{G}'_+)$ is also an algebra. The operation of differentiation on these algebras is investigated. The polynomially extended Laplace transformation and its connections with the differentiation are also studied.

1. Introduction. Recently, algebras of distributions and utradistributions with the tensor operation of multiplication were effectively used in physics (see e.g. [1]). It is not difficult to see that such algebras have to be defined on spaces of differentiable functions of infinitely many variables. These algebras have often an equivalent structure of scalar polynomials with pointwise multiplication, but the fact is not observed in the literature.

In the present paper, we would like to take this structure into consideration in a special case. For many reasons it is convenient to start with the space $\mathcal{G}'_+$ of Roumieu ultradistributions and its predual, which belong to the known classes (FS) and (DFS) (see e.g. [6]), respectively, and are nuclear; all these properties are important for our purposes. Algebras of scalar polynomials on such spaces may be described by means of projective symmetric tensor products. Thus, the symmetric tensor and multiplicative structures are equivalent in a certain sense. This fact is crucial for our investigations.

We give a description of properties of the differentiation on algebras $\mathsf{P}(\mathcal{G}'_+)$ and $\mathsf{P}'(\mathcal{G}'_+)$ by means of their tensor representations (Theorems 4.1 and 5.1). A connection between the differentiation of polynomials and the polynomially extended Laplace transformation in the form of operator calculus is described (Theorem 6.3).

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2. Tensor representations of nuclear polynomial algebras. For polynomials on infinite dimensional spaces we refer to [3]. If X, Y are locally convex (shortly: LC) complex vector spaces, then $\mathscr{L}(X,Y) := \mathscr{L}_{\mathfrak{b}}(X,Y)$ denotes the space of all continuous linear operators endowed with the topology $\mathfrak{b}$ of uniform convergence on bounded sets in X. Further, $\mathscr{L}(X) := \mathscr{L}(X,X)$ is an algebra with the operation $\circ$ of operator composition and $X' := \mathscr{L}_{\mathfrak{b}}(X,\mathbb{C})$ is the strong dual of X.

We will denote by $[[T]]$ the commutant of $T \in \mathscr{L}(X)$ and by $\langle f \mid x\rangle$ the value of $f \in X'$ at $x \in X$. Let $\mathscr{L}^n(X,\mathbb{C}) := \mathscr{L}^n_{\mathfrak{b}}(X,\mathbb{C})$ (resp. $\mathscr{L}^n_{\mathfrak{s}}(X,\mathbb{C})$) denote the space of continuous n-linear (resp. continuous n-linear symmetric) forms defined on the Cartesian product $X \times \ldots \times X$ of n copies of X.

The symbol $\otimes_{\mathfrak{p}}$ (resp. $\odot_{\mathfrak{p}}$) denotes the completion of the algebraic tensor product $\otimes$ (resp. of the symmetric tensor product $\odot$) in the projective tensor LC topology. Consider the projective tensor product $\otimes^n_{\mathfrak{p}} X'$ (resp. the projective symmetric tensor product $\odot^n_{\mathfrak{p}} X'$) of n copies of the strong dual X' and define the symmetrization projector as follows

$$\mathfrak{s}_n\colon\ \otimes^n_{\mathfrak{p}} X' \ni f_1 \otimes \cdots \otimes f_n \mapsto f_1 \odot \cdots \odot f_n := \frac{1}{n!}\sum_{\mathfrak{s}} f_{\mathfrak{s}(1)} \otimes \cdots \otimes f_{\mathfrak{s}(n)} \in \odot^n_{\mathfrak{p}} X',$$

where the sum is taken over all permutations $\mathfrak{s}$ of the set $\{1,\ldots,n\}$. Analogously, the projective tensor product $\otimes^n_{\mathfrak{p}} X$ (resp. the projective symmetric product $\odot^n_{\mathfrak{p}} X$) may be considered for the space X.

We define the LC space $\mathsf{P}_n(X)$ of n-homogeneous polynomials on X via the canonical topological linear isomorphisms $\mathsf{P}_n(X) \simeq \mathscr{L}^n_{\mathfrak{s}}(X,\mathbb{C}) \simeq (\odot^n_{\mathfrak{p}} X)'$ described in [3]. Namely, consider the following canonical embeddings:

$$\begin{aligned}&\otimes^n\colon\ X \times \ldots \times X \ni (x_1,\ldots,x_n) \mapsto x_1 \otimes \ldots \otimes x_n \in \otimes^n_{\mathfrak{p}} X,\\ &\Gamma_n\colon\ X \ni x \mapsto (x,\cdots,x) \in X \times \ldots \times X,\end{aligned}$$

and put

$$(\odot^n_{\mathfrak{p}} X)' \ni p_n \mapsto P_n := p_n \circ \otimes^n \circ \Gamma_n \in \mathsf{P}_n(X),$$

i.e.

$$P_n(x) := \langle p_n \mid \otimes^n x\rangle, \qquad \otimes^n x := (\otimes^n \circ \Gamma_n)x = x \otimes \cdots \otimes x, \qquad x \in X.$$

We call P_n so defined the *n-homogeneous polynomial* on X.

We equip $\mathsf{P}_n(X)$ with the topology $\mathfrak{b}$ on X for $n \in \mathbb{N}$ and put $\mathsf{P}_0(X) := \mathbb{C}$. The space

$$\mathsf{P}(X) := \Big\{P = \sum_{n=0}^{m} P_n\colon\ P_n \in \mathsf{P}_n(X),\, m \in \mathbb{N}\Big\},$$

endowed with the topology $\mathfrak{b}$, is called *the space of continuous polynomials on X*. Note that the space $\mathsf{P}(X)$ is a topological algebra with the scalar unit 1 and the pointwise multiplication given by

$$P(x) \cdot Q(x) := \sum_{n\in\mathbb{Z}_+} \sum_{m=0}^{n} P_m(x) \cdot Q_{n-m}(x), \qquad x \in X.$$

We will denote by $\mathsf{P}'(X)$ and $\mathsf{P}'_n(X)$ the strong duals of $\mathsf{P}(X)$ and $\mathsf{P}_n(X)$, respectively. The spaces $\mathsf{P}(X')$ and $\mathsf{P}_n(X')$ of polynomials for the dual X' and their duals are defined similarly.

The symbols $\prod_n^\bullet \odot_\mathfrak{p}^n X$ and $\sum_n^\bullet \odot_\mathfrak{p}^n X$, where the product and the sum (here and in the sequel) are taken over all $n \in \mathbb{Z}_+$, will mean the *LC Cartesian product* and the *LC direct sum* of X, respectively. In a similar way we mean the Cartesian product and the direct sum for X'.

From now we will assume that X is a nuclear (F) or (DF) LC space (see [5, 10]).

PROPOSITION 2.1. *There exist linear topological isomorphisms $\Upsilon_{X'}, \Upsilon_X$ and their linear extensions $\widetilde{\Upsilon}_{X'}, \widetilde{\Upsilon}_X$ such that*

$$\odot_\mathfrak{p}^n X' \overset{\Upsilon_{X'}}{\simeq} \mathsf{P}_n(X), \ \prod_n{}^{\bullet} \odot_\mathfrak{p}^n X' \overset{\widetilde{\Upsilon}_{X'}}{\simeq} \mathsf{P}'(X'),$$

$$\odot_\mathfrak{p}^n X \overset{\Upsilon_X}{\simeq} \mathsf{P}_n(X'), \ \sum_n{}^{\bullet} \odot_\mathfrak{p}^n X \overset{\widetilde{\Upsilon}_X}{\simeq} \mathsf{P}(X').$$

Consequently, the identity

$$\langle \mathsf{P}'(X') \mid \mathsf{P}(X') \rangle = \Big\langle \prod_n{}^{\bullet} \odot_\mathfrak{p}^n X' \Bigm| \sum_n{}^{\bullet} \odot_\mathfrak{p}^n X \Big\rangle$$

describes equivalence between the corresponding dual pairs.

If $X \looparrowright X'$ is a continuous dense embedding, then

$$\mathsf{P}(X') \looparrowright \mathsf{P}'(X')$$

is also a continuous dense embedding.

Proof. Since the topological isomorphism $\odot_\mathfrak{p}^n X' \simeq (\odot_\mathfrak{p}^n X)'$ holds for every nuclear (F) or (DF) space X [6, Th. 2.2], we have the isomorphism

$$\Upsilon_{X'} \colon \odot_\mathfrak{p}^n X' \ni f_n \mapsto F_n := f_n \circ \otimes^n \circ \Gamma_n \in \mathsf{P}_n(X).$$

For the dual pair $\langle \sum_n^\bullet \odot_\mathfrak{p}^n X' \mid \prod_n^\bullet \odot_\mathfrak{p}^n X \rangle$ the formula

$$F(x) := \sum_n F_n(x) = \Big\langle \sum_n{}^{\bullet} f_n \Bigm| \prod_n{}^{\bullet} \otimes^n x \Big\rangle, \qquad x \in X$$

induces the isomorphism

$$\widetilde{\Upsilon}_{X'} \colon \sum_n{}^{\bullet} \odot_\mathfrak{p}^n X' \ni f = \sum_n{}^{\bullet} f_n \mapsto F \in \mathsf{P}(X),$$

where $F = \widetilde{\Upsilon}_{X'}(f) = \sum_n \Upsilon_{X'}(f_n)$ acts as a linear extension of $\Upsilon_{X'} : \odot_\mathfrak{p}^n X' \mapsto \mathsf{P}_n(X)$.

Replacing above X by X', we obtain $\odot_\mathfrak{p}^n X \overset{\Upsilon_X}{\simeq} \mathsf{P}_n(X')$ and therefore (see [3])

$$\sum_n{}^{\bullet} \odot_\mathfrak{p}^n X \simeq \mathsf{P}(X').$$

Now, since the topological isomorphism $(\odot_\mathfrak{p}^n X')' \simeq \odot_\mathfrak{p}^n X$ holds for every nuclear (F) or (DF) space X [6], we have

$$\mathsf{P}_n'(X) \simeq (\odot_\mathfrak{p}^n X')' \simeq \odot_\mathfrak{p}^n X \simeq \mathsf{P}_n(X').$$

On the other hand, applying the known duality between Cartesian products and direct sums [10], we obtain

$$\mathsf{P}'(X') \simeq \Big(\sum_n{}^{\bullet} \odot_\mathfrak{p}^n X\Big)' \simeq \prod_n{}^{\bullet} \odot_\mathfrak{p}^n X' \simeq \prod_n{}^{\bullet} \mathsf{P}_n(X).$$

Hence the dual pair $\left\langle\prod_n^\bullet \odot_{\mathfrak p}^n X' \mid \sum_n^\bullet \odot_{\mathfrak p}^n X\right\rangle$ may be transformed to $\left\langle \mathsf{P}'(X') \mid \mathsf{P}(X')\right\rangle$. Due to the canonical embedding $\sum_n^\bullet \odot_{\mathfrak p}^n X' \subset \prod_n^\bullet \odot_{\mathfrak p}^n X'$, we have

$$\mathsf{P}(X) \simeq \sum_n^\bullet \odot_{\mathfrak p}^n X' \subset \prod_n^\bullet \odot_{\mathfrak p}^n X' \simeq \mathsf{P}'(X').$$

Since $\sum_n^\bullet \otimes^n x$ is a total subset in $\sum_n^\bullet \odot_{\mathfrak p}^n X$, the mapping $\widetilde{\Upsilon}_{X'}\colon\ \sum_n^\bullet \odot_{\mathfrak p}^n X' \to \mathsf{P}(X)$ can be linearly extended to the mapping

$$\widetilde{\Upsilon}_{X'}\colon \prod_n^\bullet \odot_{\mathfrak p}^n X' \ni f = \prod_n^\bullet f_n \mapsto F = \prod_n^\bullet \Upsilon_{X'}(f_n) \in \prod_n^\bullet \mathsf{P}_n(X)$$

by the formula

$$F\Big(\sum_n^\bullet \otimes^n x\Big) = \Big\langle \prod_n^\bullet f_n \Bigm| \sum_n^\bullet \otimes^n x\Big\rangle = \sum_n F_n(x), \quad F = \prod_n^\bullet F_n, \quad x \in X. \tag{1}$$

If $X \looparrowright X'$ is a continuous dense embedding, then so are the embeddings $\sum_n^\bullet \odot_{\mathfrak p}^n X \looparrowright \sum_n^\bullet \odot_{\mathfrak p}^n X'$ and $\sum_n^\bullet \odot_{\mathfrak p}^n X' \looparrowright \prod_n^\bullet \odot_{\mathfrak p}^n X'$. Consequently, we have

$$\mathsf{P}(X') \simeq \sum_n^\bullet \odot_{\mathfrak p}^n X \looparrowright \sum_n^\bullet \odot_{\mathfrak p}^n X' \looparrowright \prod_n^\bullet \odot_{\mathfrak p}^n X' \simeq \mathsf{P}'(X')$$

with the respective dense continuous embeddings. ■

Formula (1) means that $\mathsf{P}'(X')$ consists of polynomials on X.

Proposition 2.2. *The direct sum*

$$\sum_n^\bullet \odot_{\mathfrak p}^n X = \Big\{\varphi = \sum_n^\bullet \varphi_n\colon\ \varphi_n \in \odot_{\mathfrak p}^n X\Big\}$$

is an LC algebra with respect to the convolution

$$\varphi \star \psi := \sum_n^\bullet \Big(\sum_{m=0}^n \varphi_n \odot \psi_{n-m}\Big)$$

and the following mapping is an algebraic isomorphism

$$\Big\{\sum_n^\bullet \odot_{\mathfrak p}^n X, \star\Big\} \xrightarrow{\widetilde{\Upsilon}_X} \mathsf{P}(X').$$

Proof. For arbitrary $\varphi_n \in \odot_{\mathfrak p}^n X$ and $\psi_k \in \odot_{\mathfrak p}^k X$ we have

$$\varphi_n \odot \psi_k \in (\odot_{\mathfrak p}^n X) \odot (\odot_{\mathfrak p}^k X) \subset \odot_{\mathfrak p}^{n+k} X.$$

Hence the direct sum $\sum_n^\bullet \odot_{\mathfrak p}^n X$ is an algebra with respect to the convolution $\star$. By Proposition 2.1, we have $\odot_{\mathfrak p}^n X \overset{\Upsilon_X}{\simeq} \mathsf{P}_n(X')$, so the linear extension $\widetilde{\Upsilon}_X$ of Υ_X given by

$$\widetilde{\Upsilon}_X\colon \sum_n^\bullet \odot_{\mathfrak p}^n X \ni \varphi \mapsto \widetilde{\Upsilon}_X(\varphi) \in \mathsf{P}(X')$$

is the required algebraic isomorphism. ■

Now we suppose that $X \looparrowright X'$ is a continuous dense embedding. Then the convolution in $\{\sum_n^\bullet \odot_{\mathfrak p}^n X, \star\}$ can be extended to the convolution

$$f \star g := \prod_n^\bullet \Big(\sum_{m=0}^n f_n \odot g_{n-m}\Big)$$

in the Cartesian product

$$\prod_n^\bullet \odot_\mathfrak{p}^n X' = \Big\{ f = \prod_n^\bullet f_n \colon\ f_n \in \odot_\mathfrak{p}^n X' \Big\},$$

which is also a topological convolutional algebra.

Proposition 2.3. *The multiplication in $\mathsf{P}(X')$ can be uniquely extended to the multiplication in $\mathsf{P}'(X')$, given by the formula*

$$(P\cdot Q)\Big(\sum_n^\bullet \otimes^n x\Big) := \sum_n \sum_{m=0}^n P_m(x)\cdot Q_{n-m}(x)$$

for $Q = \prod_n^\bullet Q_n$, $P = \prod_n^\bullet P_n \in \prod_n^\bullet \mathsf{P}_n(X)$ and $x \in X$. Thus $\mathsf{P}'(X')$ is a topological algebra and $\widetilde{\Upsilon}_X$ uniquely extends to the following algebraic isomorphism

$$\Big\{ \prod_n^\bullet \odot_\mathfrak{p}^n X', \star \Big\} \overset{\widetilde{\Upsilon}_{X'}}{\simeq} \mathsf{P}'(X').$$

Proof. Proposition 2.1 together with Proposition 2.2 imply at once that the extended mapping

$$\widetilde{\Upsilon}_{X'} \colon \prod_n^\bullet \odot_\mathfrak{p}^n X' \ni f = \prod_n^\bullet f_n \mapsto F = \prod_n^\bullet F_n \in \mathsf{P}'(X')$$

gives the required isomorphism of algebras. ∎

In the sequel, we will define by

$$\mathscr{L}_\Gamma\Big(\sum_n^\bullet \odot_\mathfrak{p}^n X\Big) := \begin{bmatrix} \mathscr{L}(\odot_\mathfrak{p}^n X): & n = m \\ 0: & n \neq m \end{bmatrix}_{n,m\in\mathbb{Z}_+}$$

a *subalgebra of diagonal form* in $\mathscr{L}(\sum_n^\bullet \odot_\mathfrak{p}^n X)$ endowed with the topology of uniform convergence on bounded sets; analogously, for the Cartesian product

$$\mathscr{L}_\Gamma\Big(\prod_n^\bullet \odot_\mathfrak{p}^n X\Big) := \begin{bmatrix} \mathscr{L}(\odot_\mathfrak{p}^n X): & n = m \\ 0: & n \neq m \end{bmatrix}_{n,m\in\mathbb{Z}_+} \subset \mathscr{L}\Big(\prod_n^\bullet \odot_\mathfrak{p}^n X\Big).$$

We will define also diagonal subalgebras for the dual X'.

Using the isomorphisms $\mathsf{P}(X') \simeq \sum_n^\bullet \odot_\mathfrak{p}^n X$ and $\mathsf{P}'(X') \simeq \prod_n^\bullet \odot_\mathfrak{p}^n X'$ we will identify the appropriate operator algebras, namely:

$$\mathscr{L}\Big(\sum_n^\bullet \odot_\mathfrak{p}^n X\Big) \simeq \mathscr{L}(\mathsf{P}(X')), \qquad \mathscr{L}_\Gamma\Big(\sum_n^\bullet \odot_\mathfrak{p}^n X\Big) \simeq \mathscr{L}_\Gamma(\mathsf{P}(X')),$$

$$\mathscr{L}\Big(\prod_n^\bullet \odot_\mathfrak{p}^n X\Big) \simeq \mathscr{L}(\mathsf{P}(X')), \qquad \mathscr{L}_\Gamma\Big(\prod_n^\bullet \odot_\mathfrak{p}^n X'\Big) \simeq \mathscr{L}_\Gamma(\mathsf{P}'(X')).$$

We denote by $[[T]]_\Gamma$ *the commutant* in $\mathscr{L}_\Gamma(\cdot)$ of an operator $T \in \mathscr{L}_\Gamma(\cdot)$.

3. A multiplicative algebra of polynomial ultradistributions. Fix $\beta > 1$. For every $\mu > 0$ and finite $[a,b] \subset \mathbb{R}$ we define the space

$$\mathcal{G}_{[a,b]}^\mu := \{\varphi \in C^\infty(\mathbb{R})\colon\ \|\varphi\|_{\mathcal{G}_{[a,b]}^\mu} < \infty\}$$

of complex functions φ with $\operatorname{supp}\varphi \subseteq [a,b]$ and the norm

$$\|\varphi\|_{\mathcal{G}^{\mu}_{[a,b]}} := \sup_{t\in[a,b],\, k\in\mathbb{Z}_+} \frac{|D^k\varphi(t)|}{\mu^k k^{k\beta}}, \qquad D := -i\frac{d}{dt}.$$

The space $\mathcal{G}$ of Gevrey ultradifferentiable functions with compact support can be defined as the inductive limit

$$\mathcal{G} := \operatorname*{ind\,lim}_{-a,b,\mu\to\infty} \mathcal{G}^{\mu}_{[a,b]}.$$

As it is known [6, 8, 7], $\mathcal{G}$ is a nuclear (DFS)-space and is a topological algebra with respect to the pointwise multiplication.

Denote by $\mathcal{G}'$ the strong dual of $\mathcal{G}$ of all Roumieu ultradistributions on $\mathbb{R}$ [9] and let $\mathcal{G}'_+$ denote the closed subspace in $\mathcal{G}'$ of ultradistributions with supports in $[0,\infty)$. If ${\mathcal{G}'}^{\perp}_{+}$ denotes the orthogonal complement of $\mathcal{G}'_+$ with respect to $\langle \mathcal{G}' \mid \mathcal{G}\rangle$ then the factor space

$$\mathcal{G}_+ := \mathcal{G}/\,{\mathcal{G}'}^{\perp}_{+} = \{\tilde\varphi := \varphi + {\mathcal{G}'}^{\perp}_{+}\colon\ \varphi\in\mathcal{G}\}$$

is the dual of $\mathcal{G}'_+$. The operator of multiplication

$$\Theta\colon \mathcal{G}\ni\varphi\mapsto\theta\varphi\in\mathcal{G}'$$

by the Heaviside function θ has the kernel $\{\varphi\in\mathcal{G}\colon\ \operatorname{supp}\varphi\subset(-\infty,0)\} = {\mathcal{G}'}^{\perp}_{+}$. Hence, for its codomain $\Theta(\mathcal{G})$ the topological isomorphism

$$\mathcal{G}_+ \simeq \Theta(\mathcal{G})$$

holds. Thus every element $\tilde\varphi\in\mathcal{G}_+$ can be interpreted as a regular ultradistribution, belonging to $\mathcal{G}'_+$.

From duality arguments it follows that $\mathcal{G}'_+$ is a nuclear (FS)-space and $\mathcal{G}_+$ is a nuclear (DFS)-space. As it is known [2, 11], $\mathcal{G}'_+$ is a topological algebra with respect to the convolution

$$(f,g)\mapsto f*g, \qquad f,g\in\mathcal{G}'_+$$

with the Dirac function δ as the convolutional unit. Since ${\mathcal{G}'}^{\perp}_{+}$ is a closed ideal in $\mathcal{G}$, the factor space $\mathcal{G}_+$ is also a topological algebra.

PROPOSITION 3.1.

(i) *The LC algebra* $\mathsf{P}'(\mathcal{G}'_+)$ *is a* $\big\langle \mathsf{P}'(\mathcal{G}'_+) \mid \mathsf{P}(\mathcal{G}'_+)\big\rangle$*-strong completion of finite type polynomials*

$$\alpha + \sum_{n\in\mathbb{N}}\Big(\sum_{f_j\in\mathcal{G}'_+}\langle f_1\mid\tilde\varphi\rangle\cdot\ldots\cdot\langle f_n\mid\tilde\varphi\rangle\Big), \qquad \alpha\in\mathbb{C}$$

by the variable $\tilde\varphi\in\mathcal{G}_+$. *The space* $\mathcal{G}'_+$ *is closed in* $\mathsf{P}'(\mathcal{G}'_+)$.

(ii) *The LC algebra* $\mathsf{P}(\mathcal{G}'_+)$ *is a completion by uniform convergence topology on bounded sets in* $\mathcal{G}'_+$ *of finite type polynomials*

$$\alpha + \sum_{n\in\mathbb{N}}\Big(\sum_{\tilde\varphi_j\in\mathcal{G}_+}\langle f\mid\tilde\varphi_1\rangle\cdot\ldots\cdot\langle f\mid\tilde\varphi_n\rangle\Big), \qquad \alpha\in\mathbb{C}$$

by the variable $f\in\mathcal{G}'_+$. *The space* $\mathcal{G}_+$ *is closed in* $\mathsf{P}(\mathcal{G}'_+)$.

Proof. The statements follow from the topological isomorphisms $\mathsf{P}'(\mathcal{G}'_+) \simeq \prod^{\bullet}_n \odot^n_{\mathfrak{p}} \mathcal{G}'_+$ and $\mathsf{P}(\mathcal{G}'_+) \simeq \sum^{\bullet}_n \odot^n_{\mathfrak{p}} \mathcal{G}_+$, established by Proposition 2.1, with the help of additional arguments that (FS)-spaces $\odot^n_{\mathfrak{p}} \mathcal{G}'_+$ and (DFS)-spaces $\odot^n_{\mathfrak{p}} \mathcal{G}_+$ can be approximated by linear combinations of elements $f_1 \odot \cdots \odot f_n$ and $\tilde{\varphi}_1 \odot \cdots \odot \tilde{\varphi}_n$, respectively (see [4]). ∎

We will call elements of $\mathsf{P}'(\mathcal{G}'_+)$ *polynomial ultradistributions* on $[0,\infty)$. Clearly, since $\mathcal{G}'_+ \subset \mathsf{P}'(\mathcal{G}'_+)$, elements of $\mathcal{G}'_+$ can be understood as linear ultradistributions.

4. A generalized differentiation of polynomial ultradistributions. The ideal ${\mathcal{G}'}^{\perp}_+$ is invariant with respect to the right shift in the space $\mathcal{G}$, hence the diagram

$$\begin{array}{ccc} \mathcal{G}_+ \ni \tilde{\varphi} & \xrightarrow{T_t} & \tilde{\varphi}(\cdot + t) \in \mathcal{G}_+ \\ \Theta \uparrow & & \Theta \uparrow \\ \mathcal{G} \ni \varphi & \longrightarrow & \varphi(\cdot + t) \in \mathcal{G} \end{array}$$

uniquely defines a semigroup $[0,\infty) \ni t \mapsto T_t$ of the operators $T_t \in \mathscr{L}(\mathcal{G}_+)$.

The ideal ${\mathcal{G}'}^{\perp}_+$ is also invariant with respect to the differentiation D, hence D uniquely defines some differentiation D_+ on the algebra $\mathcal{G}_+$ (in Leibniz's sense), which can be also defined as a generator of semigroup T_t.

The factor topology on $\mathcal{G}^\mu_+[0,b] := \mathcal{G}^\mu_{[a,b]} \Big/ {\mathcal{G}'}^{\perp}_+ \cap \mathcal{G}^\mu_{[a,b]}$ is defined by the factor norms

$$\|\tilde{\varphi}\|_{\mathcal{G}^\mu_+[0,b]} := \sup_{t\in[a,b],\, k\in\mathbb{Z}_+} \frac{|D^k_+ \tilde{\varphi}(t)|}{\mu^k k^{\beta k}}$$

and

$$\mathcal{G}_+ \simeq \operatorname*{ind\,lim}_{b,\mu\to\infty} \mathcal{G}^\mu_+[0,b].$$

Let T'_t and $D'_+ = -D_+$ denote the corresponding adjoints with respect to $\langle \mathcal{G}'_+ \mid \mathcal{G}_+ \rangle$.

THEOREM 4.1.

(i) *The family* $\{\Gamma(T'_t) \colon t \in [0,\infty)\}$ *of continuous linear operators on* $\mathsf{P}(\mathcal{G}'_+)$ *of the form*

$$\Gamma(T'_t) \colon Q \mapsto Q \circ T'_t, \qquad Q = \sum_n Q_n \in \mathsf{P}(\mathcal{G}'_+),$$

where $Q_n = q_n \circ \otimes^n \circ \Gamma_n$ *and* $q_n \in \odot^n_{\mathfrak{p}} \mathcal{G}_+$, *acting as*

$$\Gamma(T'_t)Q(f) = Q(T'_t f) \qquad \text{for all} \qquad f \in \mathcal{G}'_+,$$

is an equicontinuous C_0*-semigroup of automorphisms on the algebra* $\mathsf{P}(\mathcal{G}'_+)$.
An equivalent tensor representation on $\sum^{\bullet}_n \odot^n_{\mathfrak{p}} \mathcal{G}_+ \simeq \mathsf{P}(\mathcal{G}'_+)$ *of its generator* $\mathsf{d}\Gamma(D'_+)$ *belongs to the subalgebra* $\mathscr{L}_\Gamma(\sum^{\bullet}_n \odot^n_{\mathfrak{p}} \mathcal{G}_+)$ *and on every element* $q = \sum^{\bullet}_n q_n$ *it acts as*

$$\mathsf{d}\Gamma(D'_+)q = \sum^{\bullet}_n \sum_{j=1}^n {}^n_j D_+ q_n, \qquad {}^n_j D_+ := \overbrace{1_+ \otimes \ldots \otimes \underbrace{D_+}_{j} \otimes \ldots \otimes 1_+}^{n}.$$

(ii) *The family* $\{\Gamma(T_t) \colon t \in [0,\infty)\}$ *of continuous linear operators on* $\mathsf{P}'(\mathcal{G}'_+)$ *of the form*

$$\Gamma(T_t) \colon P \mapsto P \circ T_t, \qquad P = \prod^{\bullet}_n P_n \in \mathsf{P}'(\mathcal{G}'_+),$$

where $P_n = p_n \circ \otimes^n \circ \Gamma_n$ *and* $p_n \in \odot_{\mathfrak{p}}^n \mathcal{G}'_+$*, acting as*

$$\Gamma(T_t)P(\tilde{\varphi}) = P(T_t\tilde{\varphi}) \qquad \text{for all} \qquad \tilde{\varphi} \in \mathcal{G}_+,$$

is an equicontinuous C_0*-semigroup of automorphisms on* $\mathsf{P}'(\mathcal{G}'_+)$ *with the generator* $\mathsf{d}\Gamma(D_+)$*, which, in an equivalent tensor representation on* $\prod_n^\bullet \odot_{\mathfrak{p}}^n \mathcal{G}'_+ \simeq \mathsf{P}'(\mathcal{G}'_+)$*, belongs to* $\mathscr{L}_\Gamma(\prod_n^\bullet \odot_{\mathfrak{p}}^n \mathcal{G}'_+)$ *and acts as*

$$\mathsf{d}\Gamma(D_+)p = -\prod_n^\bullet \sum_{j=1}^n {}_j^n D_+ p_n \qquad \text{for all} \qquad p = \prod_n^\bullet p_n.$$

(iii) *The generator* $\mathsf{d}\Gamma(D'_+)$ *is a continuous differentiation on* $\mathsf{P}(\mathcal{G}'_+)$*, that is,*

$$\mathsf{d}\Gamma(D'_+)(P \cdot Q)(f) = (\mathsf{d}\Gamma(D'_+)P \cdot Q)(f) + (P \cdot \mathsf{d}\Gamma(D'_+)Q)(f) \tag{2}$$

for all $P, Q \in \mathsf{P}(\mathcal{G}'_+)$ *and* $f \in \mathcal{G}'_+$ *(similarly, for* $\mathsf{d}\Gamma(D_+)$ *on* $\mathsf{P}'(\mathcal{G}'_+)$*).*

(iv) *The generators* $\mathsf{d}\Gamma(D_+)$ *and* $\mathsf{d}\Gamma(D'_+)$ *satisfy the dual relation*

$$\big\langle \mathsf{d}\Gamma(D_+)P \,\big|\, Q\big\rangle = -\big\langle P \,\big|\, \mathsf{d}\Gamma(D'_+)Q\big\rangle, \qquad P \in \mathsf{P}'(\mathcal{G}'_+), \quad Q \in \mathsf{P}(\mathcal{G}'_+).$$

Proof. (i) First note that the inductive limit $\mathcal{G}_+ \simeq \operatorname{ind}\lim_{\nu,b\to\infty} \mathcal{G}_+^\nu[0,b]$ has compact embeddings $\mathcal{G}_+^\mu[0,b] \looparrowright \mathcal{G}_+^\nu[0,b']$, if $\mu < \nu$, $b < b'$ [7]. Using the known property [4] that the order of inductive limits and projective tensor products can be changed, we obtain

$$\odot_{\mathfrak{p}}^n \mathcal{G}_+ \simeq \operatorname*{ind\,lim}_{\nu,b\to\infty} \odot_{\mathfrak{p}}^n \mathcal{G}_+^\nu[0,b].$$

Proposition 2.1 implies that

$$\Gamma(T'_t)Q(f) = \sum_n \big\langle q_n \,\big|\, \otimes^n T'_t f\big\rangle = \sum_n \big\langle (\otimes^n T_t) q_n \,\big|\, \otimes^n f\big\rangle$$

with the semigroup $\otimes^n T_t := T_t \otimes \ldots \otimes T_t$, acting on $\odot_{\mathfrak{p}}^n \mathcal{G}_+$. Consider $\otimes^n T_t$ on a total in $\odot_{\mathfrak{p}}^n \mathcal{G}_+^\nu[0,b]$ subset of functions $(\tau_1, \ldots, \tau_n) \mapsto \tilde{\varphi}_1(\tau_1) \odot \ldots \odot \tilde{\varphi}_n(\tau_n)$, defined on $[0,b] \times \ldots \times [0,b]$. The conditions $\tau_j \in \operatorname{supp} \tilde{\varphi}_j$ and $\tau_j - t \in \operatorname{supp}(T_t\tilde{\varphi}_j)$ are equivalent, thus,

$$\operatorname{supp}(T_t\tilde{\varphi}_j) = (\operatorname{supp} \tilde{\varphi}_j - t) \cap [0,\infty) \qquad \text{with} \qquad t \geq 0.$$

Hence,

$$\|T_t\tilde{\varphi}_j\|_{\mathcal{G}_+^\nu[0,b]} \leq \|\tilde{\varphi}_j\|_{\mathcal{G}_+^\nu[0,b]} \qquad \text{for all} \qquad \tilde{\varphi}_j \in \mathcal{G}_+^\nu[0,b], \quad t \geq 0.$$

Now, the regularity of inductive limits $\operatorname{ind}\lim_{\nu,b\to\infty} \odot_{\mathfrak{p}}^n \mathcal{G}_+^\nu[0,b]$ implies that $\otimes^n T_t$ is equibounded and, as a consequence, it is equicontinous on $\odot_{\mathfrak{p}}^n \mathcal{G}_+$. Clearly, the last conclusion uses barreledness of $\odot_{\mathfrak{p}}^n \mathcal{G}_+$ and the uniform boundedness Banach-Steinhaus principle.

Since the function

$$[0,\infty) \ni t \mapsto D_+^k \tilde{\varphi}_1(\cdot + t) \odot \ldots \odot D_+^k \tilde{\varphi}_n(\cdot + t) \in \odot_{\mathfrak{p}}^n \mathcal{G}_+$$

is smooth, the Lagrange theorem at once implies the C_0-property for $\otimes^n T_t$ on $\odot_{\mathfrak{p}}^n \mathcal{G}_+$. The equicontinuity and C_0-property on $\mathsf{P}(\mathcal{G}'_+) \simeq \sum_n^\bullet \odot_{\mathfrak{p}}^n \mathcal{G}_+$ directly follows from properties of the direct sum topology.

Since $(k_j+1)^{(k_j+1)\beta} \le 2^{(k_j+1)\beta} k_j^{k_j\beta}$, we get

$$\|D_+\tilde{\varphi}_j\|_{\mathcal{G}_+^\nu[0,b]} \le \nu \sup_{k_j\in\mathbb{Z}_+} \sup_{\tau_j\in[0,b]} \frac{|D_+^{k_j+1}\tilde{\varphi}_j(\tau_j)|}{(\nu 2^{-\beta})^{k_j+1}(k_j+1)^{(k_j+1)\beta}} \le \nu\|\tilde{\varphi}_j\|_{\mathcal{G}_+^\mu[0,b]}$$

with $\mu = \nu 2^{-\beta}$. Therefore, we have ${}^n_j D_+ \in \mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}_+)$ and the definition of a semigroup generator implies that

$$D_+(T_t\tilde{\varphi}_1 \odot \ldots \odot T_t\tilde{\varphi}_n) = \sum_{j=1}^n (\otimes^n T_t) \circ {}^n_j D_+(\tilde{\varphi}_1 \odot \ldots \odot \tilde{\varphi}_n).$$

In order to approximate an arbitrary $q \in \sum^\bullet_n \odot^n_{\mathfrak{p}}\mathcal{G}_+$ by linear combinations of $\tilde{\varphi}_1\odot\ldots\odot\tilde{\varphi}_n$ it remains to apply Proposition 3.1(ii).

The assertion (ii) follows from the duality $\langle \mathsf{P}'(\mathcal{G}'_+) \mid \mathsf{P}(\mathcal{G}'_+)\rangle = \big\langle \prod^\bullet_n \odot^n_{\mathfrak{p}}\mathcal{G}'_+ \mid \sum^\bullet_n \odot^n_{\mathfrak{p}}\mathcal{G}_+\big\rangle$ and Proposition (i).

(iii) The generator $\mathsf{d}\Gamma(D'_+)$ satisfies the equality

$$\mathsf{d}\Gamma(D'_+)Q(f) = d_f Q(D'_+f)$$

with the Fréchet derivative $d_fQ(D'_+f)$ from the polynomial $Q \in \mathsf{P}(\mathcal{G}'_+)$ at the point $f \in \mathcal{G}'_+$ by the direction D'_+f, since

$$\mathsf{d}\Gamma(D'_+)Q(T'_tf) = \frac{d}{dt}Q(T'_tf) = d_{T'_tf}Q\Big(\frac{d}{dt}T'_tf\Big) = d_{T'_tf}Q(D'_+T_tf)$$

and

$$\mathsf{d}\Gamma(D'_+)Q(f) = \mathsf{d}\Gamma(D'_+)Q(T'_tf)\mid_{t=0} = d_{T'_tf}Q(D_+T'_tf)\mid_{t=0} = d_fQ(D'_+f),$$

as a consequence. It follows that the Leibniz property (2) holds for $\mathsf{d}\Gamma(D'_+)$ and, similarly, for $\mathsf{d}\Gamma(D_+)$.

The assertion (iv) immediately follows from the dual relations

$$D'_+ = -D_+, \qquad \Big\langle p_n \,\Big|\, \sum_{j=1}^n {}^n_j D_+ q_n \Big\rangle = -\Big\langle \sum_{j=1}^n {}^n_j D_+ p_n \,\Big|\, q_n \Big\rangle$$

with $q_n \in \odot^n_{\mathfrak{p}}\mathcal{G}_+$ and $p_n \in \odot^n_{\mathfrak{p}}\mathcal{G}'_+$. ∎

5. An operator representation of differentiation. Via Theorem 4.1 for every polynomial $Q \in \mathsf{P}(\mathcal{G}'_+)$ there exists a unique $\mathsf{P}(\mathcal{G}'_+)$-value continuous function

$$Q_t\colon [0,\infty) \ni t \mapsto \Gamma(T'_t)Q \in \mathsf{P}(\mathcal{G}'_+).$$

The isomorphism $\mathsf{P}(\mathcal{G}'_+) \simeq \sum^\bullet_n \odot^n_{\mathfrak{p}}\mathcal{G}_+$ implies that for all $Q = \sum_n q_n \circ \otimes^n \circ \Gamma_n \in \mathsf{P}(\mathcal{G}'_+)$ with $q_n \in \odot^n_{\mathfrak{p}}\mathcal{G}_+$ and for all n it well-defines the unique $\odot^n_{\mathfrak{p}}\mathcal{G}_+$-value continuous function

$$[0,\infty) \ni t \mapsto (\otimes^n T_t)q_n.$$

Approaching q_n by linear combinations of $\tilde{\varphi}_1 \odot \ldots \odot \tilde{\varphi}_n \in \odot^n_{\mathfrak{p}}\mathcal{G}_+$ and using that the functions $[0,\infty) \ni t \mapsto D^k_+\tilde{\varphi}_1(\cdot+t) \odot \ldots \odot D^k_+\tilde{\varphi}_n(\cdot+t) \in \odot^n_{\mathfrak{p}}\mathcal{G}_+$ are smooth, we see that q_n belongs to $\odot^n_{\mathfrak{p}}\mathcal{G}_+ \otimes_{\mathfrak{p}} \mathcal{G}_+$. Thus $Q_t \in \mathsf{P}(\mathcal{G}'_+)\otimes_{\mathfrak{p}}\mathcal{G}_+$.

THEOREM 5.1. *The mapping (which is a polynomially extended cross-correlation)*

$$\mathsf{K}\colon \mathcal{G}'_+ \ni f \mapsto \mathsf{K}_f \in \mathscr{L}_\Gamma(\mathsf{P}(\mathcal{G}'_+)), \qquad \mathsf{K}_f(Q) := \langle f \mid Q_t\rangle, \qquad Q \in \mathsf{P}(\mathcal{G}'_+)$$

uniquely defines an algebraic topological isomorphism from the convolutional algebra $\mathcal{G}'_+$ *into the commutant* $[[\mathsf{d}\Gamma(D'_+)]]_\Gamma$ *of* $\mathsf{d}\Gamma(D'_+)$*, such that*

$$\delta \mapsto \mathsf{K}_\delta, \qquad \delta' \mapsto \mathsf{d}\Gamma(D'_+), \qquad f * g \mapsto \mathsf{K}_f \circ \mathsf{K}_g, \qquad f, g \in \mathcal{G}'_+,$$

where K_δ *is the unit in* $\mathscr{L}_\Gamma(\mathsf{P}(\mathcal{G}'_+))$*. The cross-correlation satisfies the conditions*

$$\mathsf{d}\Gamma(D'_+)(\mathsf{K}_f \circ \mathsf{K}_g) = \big[\mathsf{d}\Gamma(D'_+)\mathsf{K}_f\big] \circ \mathsf{K}_g = \mathsf{K}_f \circ \big[\mathsf{d}\Gamma(D'_+)\mathsf{K}_g\big], \qquad f, g \in \mathcal{G}'_+.$$

Proof. First note that $\langle f(t) \mid (\otimes^n T_t) q_n\rangle \in \odot^n_{\mathfrak{p}}\mathcal{G}_+$, since $(\otimes^n T_t) q_n \in \odot^n_{\mathfrak{p}}\mathcal{G}_+ \otimes_{\mathfrak{p}} \mathcal{G}_+$ and $f \in \mathcal{G}'_+$. The operator

$$\mathsf{K} : \mathcal{G}'_+ \ni f \mapsto \mathsf{K}_f \in \mathscr{L}(\mathsf{P}_n(\mathcal{G}'_+)) \simeq \mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}_+) \quad \text{with} \quad \mathsf{K}_f q_n := \langle f(t) \mid (\otimes^n T_t) q_n\rangle$$

is obviously injective and it acts as an algebraic isomorphism. In fact, the convolution in $\mathcal{G}'$ can be defined by the duality $\langle \mathcal{G}' \mid \mathcal{G}\rangle$ as follows

$$\langle f * g \mid \varphi\rangle = \langle f(t) \,\big|\, \xi(t) \langle g(s) \mid \eta(s)\varphi(t+s)\rangle\rangle$$

for any $\varphi \in \mathcal{G}$, where $\xi, \eta \in \mathcal{G}$ are 1 near $\operatorname{supp} f$ and 0 outside of supports (see [11]). We obtain

$$\begin{aligned} \mathsf{K}_{f*g} q_n &= \langle f(t) \,\big|\, \xi(t) \langle g(s) \mid \eta(s)(\otimes^n T_{t+s}) q_n\rangle\rangle \\ &= \big\langle f(t) \,\big|\, \xi(t)\mathsf{K}_g\big[\eta(\cdot)(\otimes^n T_{t+\cdot}) q_n\big]\big\rangle = (\mathsf{K}_f \circ \mathsf{K}_g) q_n. \end{aligned}$$

Thus K_δ is the unit of $\mathscr{L}[\odot^n_{\mathfrak{p}}\mathcal{G}_+]$. It also follows that

$$\mathsf{K}_{\delta'} q_n := \langle \delta'(t) \,\big|\, (\otimes^n T_t) q_n\rangle = -\Big(\sum{}_j^n D_+\Big) q_n,$$

hence, $\mathsf{K}_{\delta'}(Q) = \mathsf{d}\Gamma(D'_+)Q$. Replacing $(\sum{}_j^n D_+) q_n$ by q_n in $\mathsf{K}_f q_n = \langle f(t) \mid (\otimes^n T_t) q_n\rangle$, we obtain

$$\begin{aligned} \mathsf{K}_f\Big(\sum{}_j^n D_+\Big) q_n &= \Big\langle f(t) \,\Big|\, \Big(\sum{}_j^n D_+\Big)(\otimes^n T_t) q_n\Big\rangle \\ &= \Big(\sum{}_j^n D_+\Big)\big\langle f(t) \mid (\otimes^n T_t) q_n\big\rangle = \Big(\sum{}_j^n D_+\Big)\mathsf{K}_f q_n \end{aligned}$$

for any n, hence, $\mathsf{K}_f \in [[\mathsf{d}\Gamma(D'_+)]]$. Finally, since

$$(\delta' * f) * g = f' * g = f * g' = f * (\delta' * g),$$

we have

$$\begin{aligned} \mathsf{d}\Gamma(D'_+)(\mathsf{K}_f \circ \mathsf{K}_g)(Q) &= [\mathsf{K}_{\delta'} \circ \mathsf{K}_f] \circ \mathsf{K}_g(Q) = \mathsf{K}_{\delta' * f} \circ \mathsf{K}_g(Q) \\ &= \big[\mathsf{d}\Gamma(D_+)\mathsf{K}_f\big] \circ \mathsf{K}_g(Q) = \mathsf{K}_f \circ \mathsf{K}_{\delta' * g}(Q) \\ &= \mathsf{K}_f \circ \big[\mathsf{d}\Gamma(D'_+)\mathsf{K}_g\big](Q). \end{aligned}$$

Let us consider topological properties. The isomorphism $\mathcal{G}_+ \simeq \operatorname{ind}\lim_{\nu,b\to\infty} \mathcal{G}^\nu_+[0,b]$ implies the following adjoint isomorphism for the corresponding strong duals with a projective limit structure

$$\mathcal{G}'_+ \simeq \operatorname*{proj\,lim}_{\nu,b\to 0} (\mathcal{G}^\nu_+[0,b])'.$$

The regularity of the inductive limit $\odot^n_{\mathfrak{p}}\mathcal{G}_+ \simeq \operatorname{ind}\lim_{\nu,b\to\infty} \odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b]$ implies the embedding

$$\operatorname*{proj\,lim}_{\nu,b\to 0} \mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b]) \looparrowright \mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}_+),$$

where the operator spaces are endowed with corresponding uniform convergence topologies. By definition, every composition of K with the projector $\mathcal{G}'_+ \longrightarrow (\mathcal{G}^\nu_+[0,b])'$ acts as

$$(\mathcal{G}^\nu_+[0,b])' \ni f \mapsto f \circ J_{\mathcal{G}_+} \in \mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b]),$$

where $J_{\mathcal{G}_+} : \mathcal{G}_+ \looparrowright \odot^n_{\mathfrak{p}}\mathcal{G}_+ \otimes_{\mathfrak{p}} \mathcal{G}_+$ is the canonical continuous embedding in the projective tensor product definition. Thus it, and therefore K, are continuous. Moreover, K has a closed codomain in $\mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b])$ for all n. In fact, the equality

$$\mathcal{G}^\nu_+[0,b] = \{\langle p_n \mid (\otimes^n T_t) q_n \rangle : q_n \in \odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b],\ p_n \in (\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b])'\}$$

implies that if a sequence (f_j) is pointwise convergent to f in $\mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b])$, where $\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b]$ endowed with $\langle (\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b])' \mid \odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b] \rangle$-weak topology, then (f_j) is also weakly convergent in $\mathcal{G}^\nu_+[0,b]$. Therefore, $f \in \mathcal{G}^\nu_+[0,b]$, via the appropriate completeness. Now, the open mapping Banach theorem implies that K is a topological isomorphism from $(\mathcal{G}^\nu_+[0,b])'$ into $\mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}^\nu_+[0,b])$ for any ν, b, thus, from $\mathcal{G}'_+$ into $\mathscr{L}(\odot^n_{\mathfrak{p}}\mathcal{G}_+)$ for all n. Using a diagonal form of $\mathscr{L}_\Gamma(\mathsf{P}(\mathcal{G}'_+))$, we obtain the required isomorphism. ■

6. A polynomially extended operator calculus. By the Paley-Wiener theorem the Fourier transformation

$$\widehat{\varphi}(\zeta) := \mathcal{F}\varphi(\zeta) = \int e^{-it\zeta}\varphi(t)\,dt \qquad \text{with} \quad \varphi \in \mathcal{G},\ \zeta \in \mathbb{C},\ t \in \mathbb{R},$$

acts as a topological isomorphism

$$\mathcal{F} \colon \mathcal{G} \to \widehat{\mathcal{G}}$$

onto a space $\widehat{\mathcal{G}}$ of entire analytic functions, which we for simplicity endow with the inductive LC topology, generated by $\mathcal{F}$. In the sequel,

$$\widehat{\mathcal{G}}_+ := \widehat{\mathcal{G}}/\mathcal{F}({\mathcal{G}'}^{\perp}_{+})$$

stands for the corresponding LC factor space. For the strong duals, the appropriate adjoint transformation

$$\mathcal{F}' \colon \widehat{\mathcal{G}}' \to \mathcal{G}'$$

is defined. The codomain

$$\widehat{\mathcal{G}}'_+ := {\mathcal{F}'}^{-1}(\mathcal{G}'_+)$$

of the subspace $\mathcal{G}'_+ \subset \mathcal{G}'$ with respect to the inverse mapping

$${\mathcal{F}'}^{-1} \colon \mathcal{G}' \ni f \mapsto \widehat{f} \in \widehat{\mathcal{G}}'$$

is closed in the dual $\widehat{\mathcal{G}}'$. The mappings $\mathcal{F}'$ and ${\mathcal{F}'}^{-1}$ are continuous with respect to the strong topologies. It follows that $\widehat{\mathcal{G}}'_+$ is a nuclear (FS)-space. The space $\widehat{\mathcal{G}}'_+$ is a multiplicative topological algebra with the unit $\widehat{\delta}$, since

$$\widehat{(f * g)} = \widehat{f} \cdot \widehat{g}, \qquad f, g \in \mathcal{G}'_+.$$

A generalized Laplace transformation can be defined as

$$\mathcal{F}_+ \colon \mathcal{G}'_+ \ni f \mapsto \widehat{f} \in \widehat{\mathcal{G}}'_+, \qquad \mathcal{F}_+ := {\mathcal{F}'}^{-1}|_{\mathcal{G}'_+}. \tag{3}$$

Every element of $\widehat{\mathcal{G}}_+$ can be interpreted as the Laplace transform $\widehat{\varphi}_+ = \mathcal{F}_+(\varphi_+)$ of the regular ultradistribution

$$\varphi_+ := \Theta(\varphi) \in \mathcal{G}'_+ \qquad \text{with} \qquad \varphi \in \mathcal{G}.$$

From duality arguments it follows that the topological isomorphism

$$\mathcal{F}_+ : \mathcal{G}_+ \ni \varphi_+ \mapsto \widehat{\varphi}_+ \in \widehat{\mathcal{G}}_+$$

is true. Hence, $\widehat{\mathcal{G}}_+$ is a nuclear (DFS)-space. From Proposition 2.1 it follows that the commutative diagrams

$$\begin{array}{ccc} \mathsf{P}_n(\mathcal{G}_+) & \xrightarrow{\mathsf{F}_n} & \mathsf{P}_n(\widehat{\mathcal{G}}_+) \\ \Upsilon_{\mathcal{G}'_+}\Big\| & & \Upsilon_{\widehat{\mathcal{G}}'_+}\Big\| \\ \odot^n_{\mathfrak{p}}\mathcal{G}'_+ & \xrightarrow{\otimes^n \mathcal{F}_+} & \odot^n_{\mathfrak{p}}\widehat{\mathcal{G}}'_+, \end{array} \qquad \begin{array}{ccc} \mathsf{P}'(\mathcal{G}'_+) & \xrightarrow{\mathsf{F}_+} & \mathsf{P}'(\widehat{\mathcal{G}}'_+) \\ \widetilde{\Upsilon}_{\mathcal{G}'_+}\Big\| & & \widetilde{\Upsilon}_{\widehat{\mathcal{G}}'_+}\Big\| \\ \prod^{\bullet}_n \odot^n_{\mathfrak{p}}\mathcal{G}'_+ & \longrightarrow & \prod^{\bullet}_n \odot^n_{\mathfrak{p}}\widehat{\mathcal{G}}'_+, \end{array}$$

uniquely define the polynomial extension

$$\mathsf{F}_+ : \mathsf{P}'(\mathcal{G}'_+) \ni P = \prod_n{}^{\bullet} P_n \mapsto \widehat{P} = \prod_n{}^{\bullet} \mathsf{F}_n(P_n) \in \mathsf{P}'(\widehat{\mathcal{G}}'_+), \qquad P_n \in \mathsf{P}_n(\mathcal{G}_+)$$

of the generalized Laplace transformation (3), as an operator of the diagonal subalgebra

$$\mathscr{L}_\Gamma(\mathsf{P}'(\mathcal{G}'_+), \mathsf{P}'(\widehat{\mathcal{G}}'_+)) := \left[\begin{array}{rl} \mathscr{L}(\mathsf{P}_n(\mathcal{G}_+), \mathsf{P}_n(\widehat{\mathcal{G}}_+)) : & n = m \\ 0 : & n \neq m \end{array} \right]_{n,m\in\mathbb{Z}_+}.$$

The above diagrams and Proposition 2.2 imply that F_+ is invariant with respect to the polynomial multiplication and acts as an algebraic surjective topological isomorphism. Proposition 2.3 implies that the restriction of F_+ to $\mathsf{P}_n(\mathcal{G}'_+)$ acts also as an algebraic surjective isomorphism

$$\mathsf{F}_+ : \mathsf{P}(\mathcal{G}'_+) \ni Q = \sum Q_n \to \widehat{Q} = \sum \mathsf{F}_n(Q_n) \in \mathsf{P}(\widehat{\mathcal{G}}'_+), \qquad Q_n \in \mathsf{P}_n(\mathcal{G}'_+)$$

and the duality equivalence $\langle \widehat{P} \mid \widehat{Q} \rangle = \langle P \mid Q \rangle$ with $P \in \mathsf{P}'(\mathcal{G}'_+)$, $Q \in \mathsf{P}(\mathcal{G}'_+)$ holds.

Let us reduce the cross-correlation concept, considered in Theorem 5.1, to the case of linear ultradistributions. For this purpose we compare for every element $\varphi_+ \in \mathcal{G}_+$ the $\mathcal{G}_+$-valued function

$$T_t\varphi_+ : [0, \infty) \ni t \mapsto \varphi_+(\cdot + t) \in \mathcal{G}_+,$$

belonging to $\mathcal{G}_+ \otimes_{\mathfrak{p}} \mathcal{G}_+$. Then the cross-correlation

$$K : \mathcal{G}'_+ \ni f \mapsto K_f \in \mathscr{L}(\mathcal{G}_+), \qquad K_f\varphi_+ := \langle f(t) \mid T_t\varphi_+ \rangle\,, \qquad \varphi_+ \in \mathcal{G}_+,$$

can be expressed by the convolution

$$K_f\varphi_+ = f * \check{\varphi}_+ \qquad \text{with} \qquad \check{\varphi}_+(t) := \varphi_+(-t).$$

Proposition 6.1. *The commutative diagram*

$$\begin{array}{ccc} \mathcal{G}'_+ & \xrightarrow{K} & [[D'_+]] \\ \mathcal{F}_+\Big\downarrow & & \Big\downarrow \\ \widehat{\mathcal{G}}'_+ & \xrightarrow{\widehat{K}} & [[\widehat{D}'_+]] \end{array}$$

uniquely defines the algebraic and topological isomorphism

$$\widehat{K}\colon \widehat{\mathcal{G}}'_+ \ni \widehat{f} \mapsto \widehat{K}_{\widehat{f}} \in [[\widehat{D}'_+]] \quad \textit{such that} \quad \widehat{K}_{\widehat{\delta}} := \widehat{1}_+ \quad \textit{is the unit in} \quad \mathscr{L}(\widehat{\mathcal{G}}_+) \quad \textit{and}$$
$$\widehat{\delta}' \mapsto \widehat{D}'_+, \qquad \widehat{f}\cdot\widehat{g} \mapsto \widehat{K}_{\widehat{f}} \circ \widehat{K}_{\widehat{g}}, \qquad \widehat{f},\widehat{g} \in \widehat{\mathcal{G}}'_+,$$

which acts from the algebra $\widehat{\mathcal{G}}'_+$ *onto the commutant* $[[\widehat{D}'_+]]$ *in* $\mathscr{L}(\widehat{\mathcal{G}}_+)$, *where* $\widehat{K}_{\widehat{f}}$ *and* $\widehat{D}'_+$ *are defined as*

$$\widehat{K}_{\widehat{f}}\colon \widehat{\mathcal{G}}_+ \ni \widehat{\varphi}_+ \mapsto \widehat{K_f\varphi_+} \in \widehat{\mathcal{G}}_+$$

and

$$\widehat{D'_+\varphi_+}(\zeta) = \zeta\widehat{\varphi}_+(\zeta) - \varphi_+(0), \qquad \zeta \in \mathbb{R}.$$

Proof. Since the cross-correlation mapping K for linear ultradistributions is a special case of the mapping K for polynomial ultradistributions, Theorem 5.1 is also true if we substitute $\mathcal{G}_+$ in place of $\mathsf{P}(\mathcal{G}'_+)$. Therefore, in the statement Theorem 5.1, we can put $\mathsf{K}_f = K_f$, $\mathsf{K}_\delta = K_\delta$ and $\mathsf{d}\Gamma(D'_+) = \mathsf{K}_{\delta'} = K_{\delta'} = D'_+$.

However, we can prove more that the algebraic topological isomorphism

$$K\colon \mathcal{G}'_+ \longrightarrow [[D'_+]]$$

is surjective. To show it first note that $[[D'_+]] = [[T_t]]$ with $t \geq 0$. Let $K \in \mathscr{L}(\mathcal{G}_+)$ be an operator for which

$$(K \circ T_t)\varphi_+ = (T_t \circ K)\varphi_+ \qquad \text{with} \qquad \varphi_+ \in \mathcal{G}_+.$$

We show that there is an $f \in \mathcal{G}'_+$ such that $K = K_f$. Namely, such an f can be defined by

$$\langle f \mid \varphi_+\rangle := (K\varphi_+)(0).$$

In fact, putting $T_t\varphi_+$ instead of φ_+, we obtain

$$\begin{aligned}(K_f\varphi_+)(s) &= \langle f(t) \,\big|\, T_t\varphi_+(s)\rangle = \langle f(t) \,\big|\, T_s\varphi_+(t)\rangle \\ &= ((K \circ T_s)\varphi_+)(0) = (K\varphi_+)(s) \qquad \text{with} \qquad s \geq 0.\end{aligned}$$

Now, it is enough to calculate $\widehat{D}'_+$. Since

$$(\widehat{K_f D'_+\varphi_+})(\zeta) = \zeta(\widehat{K_f\varphi_+})(\zeta) - \langle f \mid \varphi_+\rangle,$$

we have $\widehat{D'_+\varphi_+}(\zeta) = \zeta\widehat{\varphi}_+(\zeta) - \varphi_+(0)$, if $f = \delta$. The rest follows from the fact that $\mathcal{F}_+$ realizes an algebraic topological isomorphism. ∎

From Theorem 4.1 and Proposition 6.1 it follows

Corollary 6.2. *The family* $[0,\infty) \ni t \mapsto \Gamma(\widehat{T}'_+) \in \mathscr{L}(\mathsf{P}(\widehat{\mathcal{G}}'_+))$ *of operators*

$$\Gamma(\widehat{T}'_t)\colon \widehat{Q} \mapsto \widehat{Q \circ T'_t} \qquad \textit{with} \qquad \widehat{Q} = \sum_n \mathsf{F}_n(Q_n) = \sum_n \widehat{q}_n \circ \otimes^n \circ \Gamma_n,$$

where $\widehat{q}_n := (\otimes^n\mathcal{F}_+)q_n \in \odot^n_{\mathfrak{p}}\widehat{\mathcal{G}}_+$, *acting as*

$$\Gamma(\widehat{T}'_t)\widehat{Q}(\widehat{f}) = \widehat{Q}(\widehat{T'_t f}) \qquad \textit{for all} \qquad \widehat{f} \in \widehat{\mathcal{G}}'_+,$$

is an equicontinuous C_0*-semigroup of automorphisms on* $\mathsf{P}(\widehat{\mathcal{G}}'_+)$.

A tensor representation on $\sum_n^\bullet \odot_{\mathfrak{p}}^n \widehat{\mathcal{G}}_+$ *of the semigroup generator* $\mathsf{d}\Gamma(\widehat{D}'_+)$ *belongs to* $\mathscr{L}_\Gamma(\sum_n^\bullet \odot_{\mathfrak{p}}^n \widehat{\mathcal{G}}_+)$ *and on every* $\widehat{q} = \sum_n^\bullet \widehat{q}_n$ *it acts as*

$$\mathsf{d}\Gamma(\widehat{D}'_+)\widehat{q} = \sum_n^\bullet \sum_{j=1}^n {}_j^n\widehat{D}'_+\widehat{q}_n \qquad \textit{with} \qquad {}_j^n\widehat{D}'_+ := \overbrace{\widehat{1}_+ \otimes \ldots \otimes \underbrace{\widehat{D}'_+}_{j} \otimes \ldots \otimes \widehat{1}_+}^{n}.$$

Theorem 6.3. *The mapping*

$$\widehat{\mathsf{K}}\colon \widehat{\mathcal{G}}'_+ \ni \widehat{f} \mapsto \widehat{\mathsf{K}}_{\widehat{f}} \in [[\mathsf{d}\Gamma(\widehat{D}'_+)]]_\Gamma,$$

where the operator $\widehat{\mathsf{K}}_{\widehat{f}} \in \mathscr{L}_\Gamma\big[\mathsf{P}(\widehat{\mathcal{G}}'_+)\big]$ *is defined as follows*

$$\widehat{\mathsf{K}}_{\widehat{f}}\colon \mathsf{P}(\widehat{\mathcal{G}}'_+) \ni \widehat{Q} \mapsto \widehat{\mathsf{K}_f(Q)} \in \mathsf{P}(\widehat{\mathcal{G}}'_+),$$

realizes an algebraic topological isomorphism from the algebra $\widehat{\mathcal{G}}'_+$ *onto the commutant* $[[\mathsf{d}\Gamma(\widehat{D}'_+)]]_\Gamma$ *such that* $\widehat{\mathsf{K}}_{\widehat{\delta}}$ *is the unit in* $\mathscr{L}(\mathsf{P}(\widehat{\mathcal{G}}'_+))$ *and*

$$\widehat{\delta}' \mapsto \mathsf{d}\Gamma(\widehat{D}'_+), \qquad \widehat{f}\cdot\widehat{g} \mapsto \widehat{\mathsf{K}}_{\widehat{f}} \circ \widehat{\mathsf{K}}_{\widehat{g}} \quad \textit{for all} \quad \widehat{f},\widehat{g} \in \widehat{\mathcal{G}}'_+.$$

Moreover, the commutative diagram

$$\begin{array}{ccc} \mathcal{G}'_+ & \xrightarrow{\mathsf{K}} & [[\mathsf{d}\Gamma(D'_+)]]_\Gamma \\ {\scriptstyle \mathcal{F}_+}\big\downarrow & & \big\downarrow \\ \widehat{\mathcal{G}}'_+ & \xrightarrow{\widehat{\mathsf{K}}} & [[\mathsf{d}\Gamma(\widehat{D}'_+)]]_\Gamma \end{array}$$

holds.

Proof. The statement directly follows from Theorem 5.1, Proposition 6.1 and Corollary 6.2. ∎

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